PHYS 511: Electrodynamics

Homework #3

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1. **(Jackson 2.13)**

   (a) Two halves of a long hollow conducting cylinder of inner radius b and separated by a small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2b\rho}{b^2 - \rho^2}\cos\phi\right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap.
Solution:
The electric potential follows Poisson's equation $\Delta^2\phi = \frac{\rho}{\epsilon_0}$ since, in this particular problem there is no charge in the space, it reduces to Laplace's equation $^2\phi = 0$. Since the problem entails cylindrical boundary conditions we look for solution of Laplace's equation in cylindrical coordinate system. Also, since the cylinder is long, the potential has no z dependence, we can essentially solve the potential at the bottom plane of the cylinder $z = 0$ and this solution works for every z . So the general solution of Laplace's equation in polar coordinate system is

$$u(\rho, \phi) = (C_0 \ln\rho + D_0) + \sum_n (A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n})$$

Since we expect finite solution at $\rho = 0$, $D_n = 0$ otherwise it $\rho^{-n} = \infty$ which won't satisfy boundary condition. By similar arguments $C_0 = 0$ as the solution has to be finite at $\rho = 0$ but $\ln\rho$ diverges at $\rho = 0$. So the solution reduces to, (absorbing C_n into A_n and B_n)

$$u(\rho, \phi) = D_0 + \sum_n [A_n \cos n\phi + B_n \sin n\phi]\rho^n$$

The boundary condition are, at the edge of the cylinder $\rho = b$, lets choose our coordinate system such that the right half of the cylinder from $\phi = -\frac{\pi}{2}$ to $\phi = \frac{\pi}{2}$ is at potential V_1 and the left half $\phi = \frac{\pi}{2}$ to $\phi = \frac{3\pi}{2}$ is at potential V_2

$$u(b, \phi) = \begin{cases} V_1 & \text{if } -\frac{\pi}{2} < \phi < \frac{\pi}{2} \\ V_2 & \text{if } \frac{\pi}{2} < \phi < \frac{3\pi}{2} \end{cases}$$

Now at the edge of the

$$u(b, \phi) = D_0 + \sum_{n=1}^{\infty} b^n [A_n \cos n\phi + B_n \sin n\phi]$$

Now the constant coefficient D_0 can be easily calculated by integrating both sides as

$$\int_{-\pi/2}^{3\pi/2} u(b,\phi)d\phi = \int_{-\pi/2}^{3\pi/2} D_0 d\phi + \sum_n b^n \int_{-\pi/2}^{3\pi/2} [A_n \cos n\phi + B_n \sin n\phi]d\phi$$

$$\int_{-\pi/2}^{\pi/2} u(b,\phi)d\phi + \int_{\pi/2}^{3\pi/2} u(b,\phi)d\phi = \int_{-\pi/2}^{3\pi/2} D_0 d\phi + \sum_{n=1}^{\infty} b^n \left\{ A_n \cancelto{0}{\int_{-\pi/2}^{3\pi/2} \cos n\phi d\phi} + B_n \cancelto{0}{\int_{-\pi/2}^{3\pi/2} \sin n\phi d\phi} \right\}$$

$$V_1\pi + V_2\pi = D_0 2\pi + 0$$

$$D_0 = \frac{V_1 + V_2}{2}$$

Again the coefficients B_n and A_n can be calculated by using the fact that $\{\sin\phi\}_n$ and $\{\cos\phi\}_n$ form an orthogonal set of function for integer set of n. Integrating the above expression by multiplying by $\sin m\phi$ on both sides gives

$$\int_{-\pi/2}^{3\pi/2} u(b,\phi)\sin m\phi d\phi = \sum_n b^n \left\{ A_n \cancelto{0}{\int_{-\pi/2}^{3\pi/2} \cos n\phi \sin m\phi d\phi} + B_n \int_{-\pi/2}^{3\pi/2} \sin n\phi \sin m\phi d\phi \right\}$$

$$= \sum_n B_n b^n \frac{2\pi}{2}\delta_{mn} = B_m b^m \pi$$

$$\Rightarrow \quad B_m = \frac{1}{\pi b^m} \int_{-\pi/2}^{3\pi/2} u(b,\phi)\sin m\phi d\phi$$

Similarly the coefficients A_m can be calculated as

$$A_m = \frac{1}{\pi b^m} \int_{-\pi/2}^{3\pi/2} u(b,\phi)\cos m\phi d\phi$$

Since in the given problem $u(a,\phi)$ has different values for different ϕ we get

$$A_m = \frac{1}{\pi b^m}\left[\int_{-\pi/2}^{\pi/2} u(b,\phi)\cos m\phi d\phi + \int_{\pi/2}^{3\pi/2} u(b,\phi)\cos m\phi d\phi\right]$$

$$= \frac{1}{\pi b^m}\left[V_1 \int_{-\pi/2}^{\pi/2} \cos m\phi d\phi + V_2 \int_{\pi/2}^{3\pi/2} \cos m\phi d\phi\right]$$

$$= \frac{1}{\pi b^m}\left[V_1\left(\frac{1-(-1)^m}{m}\right) + V_2\left(\frac{(-1)^m - 1}{m}\right)\right]$$

$$= \frac{1}{\pi b^m}\left[(V_1 - V_2)\left(\frac{1-(-1)^m}{m}\right)\right]$$

Wirking out the integral for B_m leads to $B_m = 0$ for all m. So the final solution becomes

$$u(\rho,\phi) = \frac{V_1 + V_2}{2} + \sum_{n=1}^{\infty} \rho^n \frac{1}{\pi b^n}(V_2 - V_2)\frac{1-(-1)^n}{n}\cos n\phi$$

$$= \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi}\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \frac{1-(-1)^n}{n}\cos n\phi$$

clearly the sum term is zero for even n, for odd n the expression is just 2/n. The closed form of the sum gives the required expression

$$u(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \arctan\left(\frac{2b\rho}{b^2 - \rho^2} \cos\phi\right)$$

This gives the potential everywhere inside the cylinder. □

(b) Calculate the surface-charge density of each half of the cylinder.
Solution:
The charge density can be simply found by finding the normal component of electric field at the surface.

$$\sigma(\phi) = \epsilon_0 \frac{\partial u(\rho, \phi)}{\partial \rho}\bigg|_{\rho=b} = \epsilon_0 \frac{V_1 - V_2}{\pi} \frac{\partial}{\partial \rho} \arctan\left(\frac{2\rho\cos\phi}{b^2 - \rho^2}\right)$$

This derivative was evaluated by using sympy to obtain

$$\sigma(\phi) = \epsilon_0 \frac{V_1 - V_2}{\pi} \frac{4b^2 \cdot 2b \cdot \cos\phi}{2b^4 + 2b^4 \cos 2\phi} = \epsilon_0 \frac{V_1 - V_2}{\pi} \frac{2\cos\phi}{b(1 + \cos 2\phi)} = \epsilon_0 \frac{V_1 - V_2}{\pi b \cos\phi}$$

For each halves we have the condition for ϕ. Subsisting the value of ϕ for each halves gives the charge density of each half. □

2. **(Jackson 2.15)**

(a) Show that the green function $G(x, y, x\prime, y\prime)$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \leq x \leq 1$, $0 \leq y \leq 1$, has an expansion

$$G(x, y, x\prime, y\prime) = 2 \sum_{n=1}^{\infty} g_n(y, y\prime) \sin(n\pi x) \sin(n\pi x\prime)$$

where $g_n(y, y\prime)$ satisfies

$$\left(\frac{\partial^2}{\partial y\prime^2} - n^2\pi^2\right) g_n(y, y\prime) = -4\pi\delta(y\prime - y) \quad \text{and} \quad g_n(y, 0) = g_n(y, 1) = 0$$

Solution:
The green's function solution to non homogeneous differential equation $\mathcal{L}h(x) = f(t)$ is a solution to homogeneous part of the differential equation with the source part replaced as delta function $\mathcal{L}h(x) = \delta(t - \xi)$. The obtained solution is $G(t, \xi)$, i.e., $\mathcal{L}G(t, \xi) = \delta(t - \xi)$. This solution corresponds to the homogeneous part only as it is independent of any source term $f(t)$. Let $G(t, \xi)$ be the solution to the differential equitation with the inhomogeneous part replaced by delta function $\delta(t - \xi)$. The green's function solution to Laplace's equation is then:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^y}{\partial y^y}\right) G(x, y; x\prime, y\prime) = -4\pi\delta(x\prime - x)\delta(y\prime - y)$$

Since we have boundary condition that $G(x' = 0) = 0$ and $G(x' = 1) = 0$ we take odd function fourier expansion of the Green's function

$$G(x, y; x\prime, y\prime) = \sum_{n=1}^{\infty} f_n(x, y; y\prime) \sin(n\pi x\prime) \tag{1}$$

Using this expression in the Laplace's equation we obtain

$$\sum_{n=1}^{\infty} \left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) f_n(x, y, y\prime) \sin(n\pi x\prime) = -4\pi\delta(x\prime - x)\delta(y\prime - y) \tag{2}$$

Completeness of the orthogonal functions $\sin(n\pi x)$ allows us to write the delta function as

$$\delta(x - x\prime) = \sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x\prime)$$

Replacing this expression in (2) we obtain

$$\sum_{n=1}^{\infty}\left(\frac{\partial^2}{\partial y\prime^2} - n^2\pi^2\right) f_n(x, y; y\prime)\sin(n\pi x\prime) = -4\pi\delta(y\prime - y)\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x\prime) \tag{3}$$

Comparing the function behavior of parameter x on LHS and RHS of (3) we obtain that the function f_n is sinusoidal. Separating out the y part of the expression into other function g_n we get

$$f_n(x, y; y\prime) = g_n(y, y\prime)\sin(4\pi x)$$

Now we can substitute this back into our green's function G in (1) we get

$$G(x, y; x\prime, y\prime) = \sum_{n=1}^{\infty} g_n(y, y\prime)\sin(n\pi x)\sin(n\pi x\prime)$$

Substituting this expression back to (2) we obtain

$$\left(\frac{\partial^2}{\partial y\prime^2} - n^2\pi^2\right) g_n(y, y\prime) = -4\pi\delta(y\prime - y) \tag{4}$$

This expression g_n also has to satisfy the boundary conditions as the complete greens function G so we have $g_n(y, 0) = 0$ and $g_n(y, 1) = 1$ as required. □

(b) Taking for $g_n(y, y\prime)$ appropriate linear combinations of $\sinh(n\pi y\prime)$ and $\cosh(n\pi y\prime)$ in the two regions $y\prime < y$ and $y\prime > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x, y; x\prime, y) = 0\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi x\prime)\sinh(n\pi y_<)\sinh(n\pi(1 - y_>))$$

where $y_<(y_>)$ is the smaller (larger) of y and $y\prime$.

Solution:

Now that we have the general expression for the green's function (4) we can divide the region into parts with $x\prime > x$ and $x\prime < x$, since in each of these cases, the source term in the differential equation is zero as the delta function is zero there $\delta(x' - x) = 0$ if $x' \neq x$ so we get

$$g_n(y, y\prime) = \begin{cases} g_< \equiv a_< \sinh(n\pi y\prime) + b_< \cosh(n\pi y\prime) & \text{if } y\prime < y \\ g_> \equiv a_> \sinh(n\pi y\prime) + b_> \cosh(n\pi y\prime) & \text{if } y\prime > y \end{cases}$$

Finding this function is, down to finding the unknown coefficients $a_<, a_>, b_<, b_>$. Applying the boundary condition $g_n(y, 0) = 0 = g_n(y, 1)$ we get

$$g_> = g_< \qquad \frac{\partial}{\partial y\prime}g_< = \frac{\partial}{\partial y\prime}g_> + 4\pi \qquad \text{if } y\prime > y$$

Now the boundary condition such that $g_>(y' = 1) = 0$ and $g_<(y' = 0) = 0$ suggests sinh functions suit the boundary condition than sin. Thus we get

$$g_n(y, y\prime) = \begin{cases} a_< \sinh(n\pi y\prime) & \text{if } y\prime < y \\ a_>[\sinh(n\pi y\prime) - \tanh(n\pi)\cosh(n\pi y\prime)] & \text{if } y\prime > y \end{cases}$$

Continuity requires that the function match at $y = y\prime$ so we have

$$a_< \sinh(n\pi y\prime) = a_>[\sinh(n\pi y\prime) - \tanh(n\pi y\prime)] \tag{5}$$

and the jump discontinuity of greens function require

$$\frac{\partial}{\partial y\prime} g_n(y_<) - \frac{\partial}{\partial y\prime} g_n(y_>) = 1 \tag{6}$$

The equations (5) and (6) give system of equation which can be solved as

$$\begin{pmatrix} \sinh(n\pi y) & -\sinh(n\pi y) + \tanh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi y) & -\cosh(n\pi y) + \tanh(n\pi)\cosh(n\pi y) \end{pmatrix} \begin{pmatrix} a_< \\ a_> \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{n} \end{pmatrix} =$$

Again solving this with simply gives

$$\begin{pmatrix} a_< \\ a_> \end{pmatrix} = -\frac{4}{n\sinh(n\pi)} \begin{pmatrix} \cosh(n\pi)\sinh(n\pi y) - \sinh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi)\sinh(n\pi y) \end{pmatrix}$$

Substituting the coefficients we get

$$g_n(y, y\prime) = \frac{4}{n\sinh(n\pi)} \times \begin{cases} \sinh(n\pi y\prime)[\sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)] & \text{if } y\prime < y \\ \sinh(n\pi y)[\sinh(n\pi)\cosh(n\pi y\prime) - \cosh(n\pi)\sinh(n\pi y\prime)] & \text{if } y\prime > y \end{cases}$$

As required the greens function is symmetric in its parameters. The symmetry is such that the expression looks exactly same if the parameter are exchanged. If we denote $y_<$ to be the minimum of $y$and $y\prime$ and similarly for $y_>$ we can write the above expression in a compact way as

$$g_n(y, y\prime) = \frac{4}{n\sinh(n\pi)} \sinh(n\pi y_<)\sinh(n\pi(1 - y_>))$$

Subsisting this to the greens function solution we get

$$G(x, y, x\prime, y\prime) = \sum_{n=1}^{\infty} \frac{8}{n\sinh(n\pi)} \sin(n\pi x)\sin(n\pi x\prime)\sinh(n\pi y_<)\sinh(n\pi(1 - y_>))$$

This is the required expression for the green's function. □