PHYS 511 : Electrodynamics

Homework #2

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1. **(Jackson 1.6)** A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance. Using Gauss' law, calculate the capacitance of

   (a) two large, flat, conducting sheets of area A, separated by small distance d
   Solution:
   Making a arbitrary Gaussian surface of area S near the surface and parallel to the surface of the large sheet we find that the electric field near the surface is

   $$E = \frac{\sigma}{2\epsilon_0}$$

   Since between the places both plates have the same field they add up to twice the value. Since the field is uniform between the plates, the potential difference is simply the product of the field and the separation thus we get

   $$V = Ed = 2 \times \frac{\sigma}{2\epsilon_0} \times d = \frac{\sigma d}{\epsilon_0}$$

   Also the total charge in the entire surface is simply the product the charge density and its area thus we get

   $$V = \frac{qd}{A\epsilon_0} \implies C = \frac{q}{V} = \frac{A\epsilon_0}{d}$$

   This gives the capacitance of the two large flat plates. □

   (b) two concentric conducting spheres with radii a , b $(b > a)$;
   Solution:
   For two conducting sphere, the electric field due to outer sphere in the space between two spheres is zero. The only field is due to the charge on inner conductor. Constructing a spherical Gaussian surface enclosing the inner sphere we find the total field in the region between the two spheres is

   $$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$$

   where $a < r < b$ is the distance from the center of the spheres. Now the potential difference between the spheres is the work done on unit charge moving from inner sphere to the out sphere thus we have

   $$V = \int_a^b \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab}$$

   The capacitance is now simply the ratio of Q and V which is

   $$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}$$

   This gives the capacitance of spherical capacitor. □

(c) two concentric conducting cylinders of length L , large compared to their radii a , b $(b > a)$.
Solution:
Similar to part (1b) we get no field inside the inner cylinder and outside the outer cylinder. In the space between the two, only the inner cylinder contributes to the electric field. Again with a cylindrical Gaussian surface bounding the inner cylinder we find that the field in the space between those is

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\epsilon_0} \implies E 2\pi r L = \frac{Q}{\epsilon_0} \implies E = \frac{Q}{2\epsilon_0 \pi L r}$$

where $a < r < b$ is the radial distance from the center of the cylinders. The potential difference now is again the work done on unit charge which is

$$V = \int_a^b \frac{Q}{2\epsilon_0 \pi L r} dr = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

The capacitance is by definition found by the ratio of Q to V;

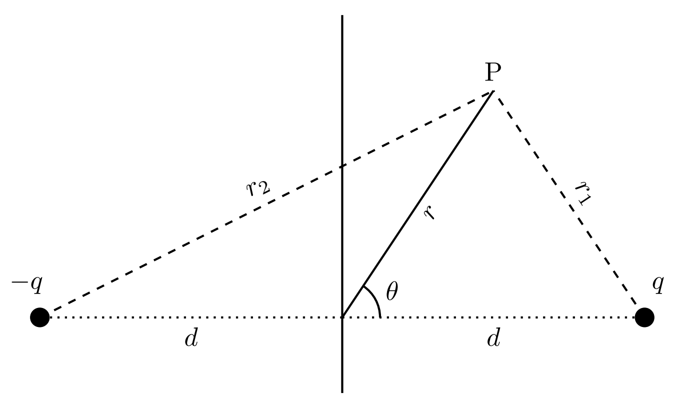
$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

This gives the capacitance of cylindrical capacitor. □

(d) What is the inner diameter of the outer conductor in an air filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m.

2. **(Jackson 2.1)** A point charge of q is brought to a position a distance d away from an infinite plane conductor held at zero potential . Using the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;
Solution:
The image charge for a point charge near the infinite conductor is behind the plane at a equal distance and the charge is of equal magnitude and opposite sign. Thus the total potential due to the image charge and the point charge (in polar coordinate system) is Using the cosine law, the different quantities



in the given diagram can be written as

$$\begin{aligned} r_1^2 &= r^2 - 2rd\cos\theta + d^2; && \implies && r_1 = \sqrt{1 - 2dr\cos\theta + d^2} \\ r_2^2 &= r^2 - 2rd\cos(\pi - \theta) + d^2; && \implies && r_2 = \sqrt{1 + 2dr\cos\theta + d^2} \end{aligned}$$

The potential at any general point $P(r, \theta)$ is given by

$$\phi(r, \theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Since by gauss's law near a conductor the surface charge density and the normal component of the field are related by the equation

$$E = \frac{\sigma}{\epsilon_0} \implies \sigma = \epsilon_0 E$$

we calculate the gradient of the potential and evaluate at its surface. The gradient is

$$\boldsymbol{E} = \frac{\partial \phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\boldsymbol{\theta}}$$

Evaluating the radial and azimuthal component and evaluating at the surface which corresponds to $\theta = \pi/2$ we get,

$$E_r = \frac{q}{4\pi\epsilon_0}\left[-\frac{-d\cos(\theta) - r}{(d^2 + 2dr\cos(\theta) + r^2)^{\frac{3}{2}}} + \frac{d\cos(\theta) - r}{(d^2 - 2dr\cos(\theta) + r^2)^{\frac{3}{2}}}\right]_{\theta=\frac{\pi}{2}} = 0$$

$$E_\theta = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[-\frac{dr\sin(\theta)}{(d^2 + 2dr\cos(\theta) + r^2)^{\frac{3}{2}}} - \frac{dr\sin(\theta)}{(d^2 - 2dr\cos(\theta) + r^2)^{\frac{3}{2}}}\right]_{\theta=\frac{\pi}{2}} = -\frac{2qd}{4\pi\epsilon_0(d^2 + r^2)^{3/2}}$$

Thus the total charge density is given by

$$\sigma = \epsilon_0 E = -\frac{qd}{2\pi(d^2 + r^2)^{3/2}}$$

This gives the required surface charge density. □

(b) the force between the charge and its image
Solution:
Since the image charge and the point charge are equal and opposite and magnitude and are a total distance $2d$ apart we get the force by columns' law as

$$F = -\frac{q^2}{4\pi\epsilon_0(2d)^2} = -\frac{q^2}{16\pi\epsilon_0 d^2}$$

This is the required force. □

(c) the total force acting on the plane by integrating $\frac{\sigma^2}{2\epsilon_0}$ over the whole plane;
Solution:
Lets assume a small circular area element at a distance r from the center of the circle then the area element is $da = 2\pi r dr$ thus the total area integral over the whole area is

$$F = \int_{r=0}^{\infty} \frac{\sigma^2}{2\epsilon_0} 2\pi r dr = \frac{qd^2}{4\pi\epsilon_0}\int_0^{\infty} \frac{r}{(r^2 + d^2)^3} dr = \frac{qd^2}{4\pi\epsilon_0} \cdot \frac{1}{4d^4} = \frac{q^2}{16\pi\epsilon_0 d^2}$$

This gives the same force as in the previous part. □

(d) the work necessary to remove the charge q from its position to infinity;
Solution:
With the image charge at d from the surface we have to move the charge from d above the surface to infinity, the total work done is given by

$$W = \int F dz = \int_{z=d}^{\infty} \frac{-q^2}{4\pi\epsilon_0(d+z)^2} dr = -\frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{d+z}\right]_d^{\infty} = -\frac{q^2}{8\pi\epsilon_0 d}$$

This is the required work for the removal of charge to infinity. □

(e) the potential energy between the charge q and its image.
Solution:
The total potential between the charge and image is simply the electric potential of two equal and opposite point charge q at a distance $2d$ thus we get

$$V = -\frac{q^2}{4\pi\epsilon_0(2d)} = \frac{-q^2}{8\pi\epsilon_0 d}$$

This is the potential between the charge. As required, this is exactly the same as we got in (2d). □

(f) Find the answer to part (2d) in electron volts for an electron originally one angstrom from the surface.
Solution:
For $d = 1 \times 10^{-10}$ and $q = 1e - 19$ and $\epsilon_0 = 8.85 \times 10^{-12}$ we get

$$V = -\frac{q^2}{8\pi\epsilon d} = 1.15 \times 10^{-18} J = 7.19 eV$$

Thus the potential energy between the charges is $7.19eV$. □

3. **(Jackson 2.7)** Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity),

(a) Write down the appropriate Green function $G(\boldsymbol{x}, \boldsymbol{x'})$,
Solution:
Let there be a point charge q $\boldsymbol{x'} = (\rho', \phi', z')$. For the potential to be zero at plane $z = 0$ we assume a image charge $-q$ at $(\rho', \phi', -z')$. The green's function is simply the potential due to these point charge at a general location $\boldsymbol{x} = (\rho, \phi, z)$

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{r_1} - \frac{1}{r_2}$$

where r_1 is the distance of general point to the point charge and r_2 is the distance from image charge to the general point. We can calculate the distances as

$$r_1 = \sqrt{(\rho\cos\phi - \rho'\cos\phi)^2 + (z - z')^2 + (\rho\sin\phi - \rho\sin\phi')^2} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z - z')^2}$$

Similarly

$$r_2 = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z + z')^2}$$

Since the choice of coordinate system is arbitrary due to azimuthal symmetry, we can choose $\phi' = 0$ without loss of generality.

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{r_1} - \frac{1}{r_2} = \left[\frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi + (z - z')^2}} - \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi + (z + z')^2}}\right]$$

This is the greens' function. □

(b) if the potential on the place $z = 0$ is specified to be $\Phi = V$ inside a circle of radio's a centered at the origin, and $\Phi = 0$ outside that circle, Find integral expression for the potential ant he point P specified in terms of cylindrical coordinates (ρ, ϕ, z).
Solution:
The integral equation to solve for the potential is

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int_V G(\boldsymbol{x}, \boldsymbol{x'})\rho(x')dV - \frac{1}{4\pi}\oint_s \Phi(\boldsymbol{x'})\frac{\partial G(\boldsymbol{x}, \boldsymbol{x'})}{\partial n}da'$$

Since we don't have charge density inside the volume bounded by the cylinder $\rho(\boldsymbol{x'}) = 0$ thus the only remaining term is the second term. The outward normal on the surface of the cylinder can be calculated. But since the potential at upper infinite plane is zero it has no contribution. Similarly the sidewall of the cylinder do not contribute to the integral because the cylindrical wall have a surface area infinity and the integral goes to zero. Thus the only contribution comes from base of cylinder with radius a . On this face $z' = 0$ so we get

$$\begin{aligned}\left.\frac{\partial G}{\partial n}\right|_{z'=0} &= \left[\frac{1}{2}\frac{2(z-z')}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z-z')^2)^{3/2}} + \frac{1}{2}\frac{2(z+z')}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z+z')^2)^{3/2}}\right]_{z=0} \\ &= \frac{2z}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+z^2)^{3/2}}\end{aligned}$$

Since the potential at that surface is $\Phi(\boldsymbol{x'}) = V$ we get

$$\Phi = \frac{V}{4\pi}\int\limits_{\rho'=0}^{a}\int\limits_{\phi'=0}^{2\pi}\frac{2z}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+z^2)^{3/2}}\rho'\,d\phi'\,d\rho'$$

This is the required integral expression. □

(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Solution:
Solving at $\rho = 0$ we get

$$\Phi(x) = \frac{2Vz}{4\pi}\cdot 2\pi\int\limits_0^a\frac{\rho'}{(\rho'^2+z^2)^{3/2}}d\rho' = Vz\left[\frac{1}{\sqrt{\rho'^2+z^2}}\right]_0^a = Vz\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+a^2}}\right] = V\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]$$

Which is the required expression. □