PHYS:502 Mathematical Physics II

Homework #7

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1. Show that

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{r'^l}{r^{l+1}}\right) \frac{4\pi}{2l+1} Y_l^m(\theta, \phi)^* Y_l^m(\theta', \phi')$$

Solution:
Let the angle between the vectors $\mathbf{r}'$ and $\mathbf{r}$ be γ. Also let $|\mathbf{r} - \mathbf{r}'| = r_1$. Then by cosine law wwe have

$$r_1^2 = r'^2 - 2r'r\cos\gamma + r^2;$$

Which can be rearranged to get

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r_1} = \frac{1}{r}\left(\frac{r'}{r} - 2\frac{r'}{r}\cos\gamma + 1\right)^{-1/2} = \frac{1}{r}\sum_{l=0}^{\infty} P_l(\cos\gamma)\left(\frac{r'}{r}\right)^l$$

From the spherical harmonics addition theorem we can write

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta, \phi)^* Y_l^m(\theta', \phi')$$

Substuting this in the above expression we get

$$\begin{aligned}\frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta, \phi)^* Y_l^m(\theta', \phi') \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r'^l}{r^{l+1}}\right)\frac{4\pi}{2l+1} Y_l^m(\theta, \phi)^* Y_l^m(\theta', \phi')\end{aligned}$$

Clearly this series converses only if $r > r'$ if instead $r' > r$ in the the expression can be rewritten as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r_1} = \frac{1}{r'}\left(1 - 2\frac{r}{r'}\cos\gamma + \frac{r}{r'}\right)^{-1/2} = \frac{1}{r'}\sum_{l=0}^{\infty} P_l(\cos\gamma)\left(\frac{r}{r'}\right)^l$$

Using the spherical harmonics addition relation leads to

$$\begin{aligned}\frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta, \phi)^* Y_l^m(\theta', \phi') \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r^l}{r'^{l+1}}\right)\frac{4\pi}{2l+1} Y_l^m(\theta, \phi)^* Y_l^m(\theta', \phi')\end{aligned}$$

These are the required expressions ■

2. By choosing a suitable form for h in the generating function

$$G(z,h) = \exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) h^n$$

show that the integral representation of the bessel functions of the first kind are given, for integral m by

$$J_{2m}(z) = \frac{(-1)^m}{2\pi}\int_0^{2\pi} \cos(z\cos\theta)\cos(2m\theta)d\theta; \qquad m \geq 1,$$

$$J_{2m+1}(z) = \frac{(-1)^m}{2\pi}\int_0^{2\pi} \sin(z\cos\theta)\cos((2m+1)\theta)d\theta \qquad m \geq 0.$$

Solution:
Let $h = ie^{i\theta}$. With this choice of h we get $h - 1/h = ie^{i\theta} + ie^{-i\theta} = 2i\cos\theta$. This siimplifies the generating function integral to

$$\begin{aligned}
e^{iz\cos\theta} &= \sum_{n=-\infty}^{\infty} J_n(z)\left(ie^{i\theta}\right)^n \\
\cos(z\cos\theta) + i\sin(z\cos\theta) &= \sum_{n=-\infty}^{\infty} J_n(z) i^n (\cos\theta + i\sin\theta)^n \\
&= \sum_{n=-\infty}^{\infty} i^n J_n(z)\cos n\theta + i^{n+1} J_n(z)\sin n\theta
\end{aligned}$$

Since i^n is real for even n and i^{n+1} is real for odd n. The real part of the expression on RHS is

$$\sum_{m=-\infty}^{\infty} J_{2m} i^{2m}\cos(2m\theta) + J_{2m+1} i^{2m+2}\sin((2m+1)\theta) = \sum_{m=-\infty}^{\infty} J_{2m}(-1)^m\cos(2m\theta) + J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)$$

Thus equating real part on both sides gives

$$\cos(z\cos\theta) = \sum_{m=-\infty}^{\infty} J_{2m}(-1)^m\cos(2m\theta) + J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)$$

Since we know that the set $\{\sin n\theta\}_n$ and $\{\cos n\theta\}_n$ form orthogonal set of functions we can find the expression J_{2m} by usual "Fourier Trick" as

$$\begin{aligned}
\int_0^{2\pi} \cos(z\cos\theta)\cos(2r\theta)d\theta &= \int_0^{2\pi}\left(\sum_{m=-\infty}^{\infty} J_{2m}(-1)^m\cos(2r\theta) + J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)\right)\cos 2r\theta d\theta \\
&= \sum_{m=-\infty}^{\infty}\left(\int_0^{2\pi} J_{2m}(-1)^m\cos(2m\theta)\cos 2r\theta d\theta + \int_0^{2\pi} J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)\cos 2r\theta d\theta\right) \\
&= \sum_{m=-\infty}^{\infty} (-1)^m J_{2m} 2\pi\delta_{mr} + 0 \\
&= (-1)^r J_{2r}(z) 2\pi
\end{aligned}$$

Rearraning the expression gives since $\frac{1}{(-1)^r} = (-1)^r$

$$J_{2r}(z) = \frac{(-1)^r}{2\pi}\int_0^{2\pi} \cos(z\cos\theta)\cos(2r\theta)d\theta$$

Similarly equating the imaginary part gives

$$\sin(z\cos\theta) = \sum_{m=-\infty}^{\infty} J_{2m}(-1)^m \sin(2m\theta) + J_{2m+1}(-1)^{m+1}\cos((2m+1)\theta)$$

The usual orthogonality gives

$$J_{2r+1}(z) = \frac{(-1)^r}{2\pi}\int\limits_0^{2\pi} \sin(z\cos\theta)\cos((2r+1)\theta)d\theta$$

These concludes the rquirement. ■

3. Find the potential distribution in a hollow conducting cylinder of radius R and length l. The two ends are closed by conducting plates. One end of the plate and the cylindrical wall are held at potential $\Phi = 0$. The other end plate is insulated form the ylindre and held ate potential $\Phi = \phi_0$
Solution:
Since there is no charge source inside the cylinder, the potential in a chargeless region follows the lapalces equation $\nabla^2\Phi = 0$. Using the usual culindrical coordinate system for the problem the general solution of the Laplaces equation in cylindrical solution is given by

$$\Phi(\rho,\phi,z) = [AJ_m(k\rho) + BY_m(k\rho)][C\cos m\phi + D\sin m\phi]\big[Ee^{-kz} + Fe^{kz}\big]$$

Since the potential is finite at $\rho = 0$ at the axis of cylinder, the coefficient $B = 0$ because $Y_m(0) = -\infty$. Since the potential is finite in that region that has to be the case. Also since there is azimuthal symmetry the value of $m = 0$. The general solution then becomes

$$\Phi(\rho,\phi,z) = AJ_0(k\rho)\big[Ee^{-kz} + Fe^{-kz}\big]$$

Since the potential is 0 at $z = 0$ in the bottom end of cylinder. $E + F = 0; \qquad E = -F$. Absorbing $2F$ into A we get

$$\Phi(\rho,\phi,z) = AJ_0(k\rho)\sinh(kz)$$

Also at the wall of the cylinder $\rho = a$ the potential is zero so

$$0 = \Phi(a,\phi,z) = AJ_0(ka)\sinh(kz)$$

The only way this expression can be zero for all z is if $J_0(ka) = 0$. Which means ka should be the zero of bessel function. Since there are infinite zeros of bessel functions let them be denoted by $\{\alpha_i\}_{i=0}^{\infty}$. This means $ka = \alpha_i; \Rightarrow k_i = \frac{\alpha_i}{a}$ So the general solution becomes

$$\Phi(\rho,\phi,z) = \sum_{i=0}^{\infty} A_i J_0\Big(\frac{\alpha_i}{a}\rho\Big)\sinh\Big(\frac{\alpha_i}{a}z\Big)$$

The coefficient A_i is given by

$$A_i = \frac{2}{J_1^2(\alpha_i)\sinh\big(\frac{\alpha_i}{a}l\big)}\int_0^a \rho\Phi(\rho,\phi,l)J_0\Big(\frac{\alpha_i}{a}\rho\Big)d\rho$$

Since $\Phi(\rho,\phi,l) = \phi_0$ this integral is becomes

$$\begin{aligned}
A_i &= \frac{2\phi_0}{J_1^2(\alpha_i)\sinh\big(\frac{\alpha_i}{a}l\big)}\int_0^a \rho J_0\Big(\frac{\alpha_i}{a}\rho\Big)d\rho \\
&= \frac{2\phi_0}{J_1^2(\alpha_i)\sinh\Big(\frac{\alpha_i)}{a}l\Big)}\left[\frac{J_1(\alpha_i)}{\alpha_i}\right] \\
&= \frac{2\phi_0}{\alpha_i J_1(\alpha_i)\sinh\big(\frac{\alpha_i}{a}l\big)}
\end{aligned}$$

Substuting this back gives the required general solution

$$\Phi(\rho,\phi,z)=\sum_{i=0}^{\infty}\frac{2\phi_0}{\alpha_i J_1(\alpha_i)\sinh\left(\frac{\alpha_i}{a}l\right)}J_0\left(\frac{\alpha_i}{a}\rho\right)\sinh\left(\frac{\alpha_i}{a}z\right)$$

This gives the potential everywhere inside the cylinder. ■

4. Show from its definition, that the Bessel function of second kind, and of integer order ν can be written as

$$Y_\nu(z)=\frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial\mu}-(-1)^\nu\frac{\partial J_{-\mu}(z)}{\partial\mu}\right]_{\mu=\nu}$$

Using the explicit series expression for $J_\mu(z)$, show that $\partial J_\mu(z)/\partial\mu$ can be written as

$$J_\nu(z)\ln\left(\frac{z}{2}\right)+g(\nu,z)$$

and deduce that $Y_\nu(z)$ can be expressed as

$$Y_\nu(z)=\frac{2}{\pi}J_\nu(z)\ln\left(\frac{z}{2}\right)+h(\nu,z)$$

Where $h(\nu,z)$ lik $g(\nu,z)$, is a power series in z.
Solution:
The definition of the bessel function of second kind is

$$Y_\nu(z)=\lim_{\mu\to\nu}\frac{\cos\mu\pi J_\mu(z)+J_{-\mu}(z)}{\sin\mu\pi}$$

Using L Hopitals rule to evaluate this limit we get

$$Y_\nu(z)=\lim_{\mu\to\mu}\frac{-\pi\sin\mu\pi J_\mu(z)+\cos\mu\pi J'_\mu(z)-(-1)^\mu J'_{-\mu}(z)}{\cos\mu\pi}$$

Since at integer values of ν the value $\cos\nu\pi=1$ and $\sin\nu\pi=0$ we get

$$Y_\nu(z)=\frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial\mu}-(-1)^\nu\frac{\partial J_{-\mu}(z)}{\partial\mu}\right]_{\mu=\nu}$$

For non-integer ν the power series representation of the Bessel function is

$$J_\mu(z)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}$$

Taking derivative with respect to μ we get

$$\begin{aligned}\frac{\partial J_\mu(z)}{\partial\mu}&=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}\ln\left(\frac{z}{2}\right)+\sum_{r=0}^{\infty}-\frac{(-1)^r\Gamma'(r+\mu+1)}{r!\Gamma^2(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}\\&=\ln\left(\frac{z}{2}\right)\underbrace{\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}}_{J_\mu(z)}+\underbrace{\sum_{r=0}^{\infty}-\frac{(-1)^r\Gamma'(r+\mu+1)}{r!\Gamma^2(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}}_{g(\mu,z)}\\&=\ln\left(\frac{z}{2}\right)J_\mu(z)+g(\mu,z)\end{aligned}$$

Since $J_{-\mu}(z) = (-1)^\mu J_\mu(z)$. This expression can be reused to calculate the derivative of $J_{-\mu}$. Multiplying both sides of this expression by $(-1)^\mu$ we get

$$\frac{\partial J_{-\mu}(z)}{\partial \mu} = (-1)^\mu \ln\left(\frac{z}{2}\right) J_\mu(z) + (-1)^\mu g(\mu, z)$$

Substuting this back in the expression for the bessel function of second kind we get

$$\begin{aligned} Y_\nu(z) &= \frac{1}{\pi}\left[\ln\left(\frac{z}{2}\right) J_\mu(z) + g(\mu, z) + (-1)^\mu (-1)^\mu \ln\left(\frac{z}{2}\right) J_\mu(z) + (-1)^\mu g(\mu, z)\right]_{\mu=\nu} \\ &= \frac{1}{\pi}\left[\ln\left(\frac{z}{2}\right) J_\nu(z) + \ln\left(\frac{z}{2}\right) J_\nu(z)\right] + \underbrace{\frac{1}{\pi}[g(\nu, z) + (-1)^\nu g(\nu, z)]}_{h(\nu,z)} \\ &= \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_\nu(z) + h(\nu, z) \end{aligned}$$

This gives the requried expression for Bessel function of second kind for integer order. ■