PHYS :502 Mathematical Physics II

Homework #6

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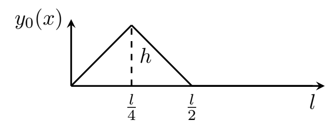
1. A string fixed at both ends and of length l has a zero initial velocity and an initial displacement as shown in the figure. Find the subsquesnt displacement of the string as a function of x and t.



Solution:
The motion of the string is guided by the wave equation which can be written as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

If we suppose the solution $y(x,t) = X(x)T(t)$ then substuting these and dividing thourgh by XT we obtain

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

The above solution is composed of two parts, each function of independent variables, the only way they can be equal is if they are equal to constant, let the constant that they are equal be k^2.

$$\frac{X''}{X} = k^2; \qquad \Rightarrow X = A\sin(kx) + B\cos(kx)$$
$$\frac{1}{c^2}\frac{T''}{T} = k^2; \qquad \Rightarrow T = D\sin(kct) + E\cos(kct)$$

So the solution to the differential equation becomes,

$$u(x,t) = [A\sin(kx) + B\cos(kx)][D\cos(kcx) + E\sin(kcx)] \tag{1}$$

But since the string is stationary at both ends. At $x = 0$ and $x = L$

$$0 = B\cos(kx)[D\cos(kct) + E\sin(kct)]$$

The only way it can be zero for all t is if $B = 0$. Substuting $B = 0$ in (1) and differentiating with respect to t.

$$\frac{\partial}{\partial t}u(x,t) = A\sin(kx)[kc(-D\sin(kcx) + E\cos(kcx))]; \qquad \Rightarrow u'(x,0) = 0 = A\sin(kx)[Ekc]$$

The only way the above expression can be zero for all x is if $E = 0$. Also since $u(l,t) = 0$ for all t, the only way this can happen is if $k = \frac{n\pi}{l}$. Since we have different possible values of n for solution, the linear combination of all will be the most general solution

$$u(x,t) = \sum_n A_n \sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}ct\right)$$

The shape of the string at he begining is given as a part function. So the function at $t = 0$ then becomes

$$u(x,0) = \sum_n A_n \sin\left(\frac{n\pi}{l}x\right)$$

The coefficients A_n can be found by usual "Fourier Trick" as

$$A_n = \frac{2}{l}\int_0^l u(x,0)\sin\left(\frac{n\pi}{l}x\right)dx$$

Since the given velocity function is two part function we obtain A_n as

$$\begin{aligned}
A_n &= \frac{2}{l}\left[\int_0^{l/4} u(x,0)\sin\left(\frac{n\pi}{l}x\right)dx + \int_{l/4}^{l/2} u'(x,0)\sin\left(\frac{n\pi}{l}x\right)dx + \int_{l/2}^{l} u(x,0)\sin\left(\frac{n\pi}{l}x\right)dx\right] \\
&= \frac{2}{l}\left[\int_0^{l/4} \frac{2h}{l}x\sin\left(\frac{n\pi}{L}x\right)dx + \int_{l/4}^{l/2} -\frac{4h}{l}\left(x - \frac{l}{2}\right)\sin\left(\frac{n\pi}{l}x\right)dx\right] \\
&= \frac{8h}{\pi^2 n^2}\left(2\sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{\pi n}{2}\right)\right)
\end{aligned}$$

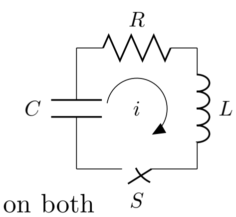
Substiting this back into the solution we have

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8h}{\pi^2 n^2}\left(2\sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{\pi n}{2}\right)\right)\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{n\pi}{l}ct\right)$$

This gives the position of every point in the string as a function of time. ■

2. A RLC circuit has the charge stored in capacitor q which satisfies the differential equation as

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0$$



If the charge in the capacitor at time $t = 0$ is $q(0) = q_0$ find the charge as a function of time.

Solution:
Let us assume that the laplace transform of $q(t)$ is $\mathcal{L}\{q(t)\} = Q(s)$. Taking laplace transform on both sides we get

$$\begin{aligned}
L\{s^2 Q(s) - sq(0) - q'(0)\} + R\{sQ(s) - q(0)\} + \frac{1}{C}Q(s) = 0 \\
\left(Ls^2 + Rs + \frac{1}{C}\right)Q(s) - (Ls + R)q(0) - Lq'(0) = 0
\end{aligned}$$

$$Q(s) = \frac{L\left(s + q'(0) + \dfrac{R}{L}\right)}{L\left(s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}\right)} = \frac{s}{\left(s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}\right)} + \frac{q'(0) + \dfrac{R}{L}}{\left(s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}\right)}$$

Since there is charge in the capacitor. The initial rate of discharge of capacitor is $q'(0) = \frac{R}{L}$. The denominator can be written as a complete square sum and the expression becomes

$$Q(s) = \frac{s}{\left(s + \dfrac{R}{2L}\right)^2 + \left(\dfrac{1}{LC} - \dfrac{R^2}{4L^2}\right)} + \frac{q'(0) + \dfrac{R}{L}}{\left(s + \dfrac{R}{2L}\right)^2 + \left(\dfrac{1}{LC} - \dfrac{R^2}{4L^2}\right)}$$

Writing $\frac{R}{2L} = \alpha$ and $\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right) = \omega^2$ we get

$$Q(s) = \frac{s + \alpha}{(s+\alpha)^2 + \omega^2} + \frac{\alpha}{\omega}\frac{\omega}{(s+\alpha)^2 + \omega^2}$$

The inverse lapalace transform gives

$$q(t) = e^{-\alpha t}\Big(\sin(\omega t) + \frac{\alpha}{\omega}\cos(\omega t)\Big)$$

This is the required charge as a function of time in the capacitor. ■

3. Solve the diffusion equation $\frac{\partial^2}{\partial x^2}q(x,t) = \frac{\partial}{\partial t}q(x,t)$ for the initial condition. $q(x,0) = n_0 e^{-\alpha|x|}$
Solution:

$$\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial q(x,t)}{\partial t}$$

Taking fourier transform in variable x on both sides

$$\int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}q(x,t) = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}q(x,t); \qquad \Rightarrow -k^2 Q(k,t) = \frac{\mathrm{d}}{\mathrm{d}t}Q(k,t)$$

This is a first order differential equation in t whicch has a solution

$$Q(k,t) = A_0 e^{-k^2 t}$$

Now given the boundary condition $q(x,0) = e^{-\alpha|x|}$ we can calculate the contant A_0 by

$$\begin{aligned}
Q(k,0) = \mathcal{F}(q(x,0)) &= \int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-ikx}dx \\
&= \int_{-\infty}^{0} e^{\alpha x}e^{-ikx}dx + \int_{0}^{\infty} e^{-\alpha x}e^{-ikx}dx \\
&= \int_{-\infty}^{0} e^{(\alpha-ik)x}dx + \int_{0}^{\infty} e^{-(\alpha+ik)x}dx \\
&= \frac{n}{\alpha-ik}e^{\alpha-ik}\Big|_{-\infty}^{0} - \frac{n_0}{x(\alpha+ik)}e^{-x(\alpha-ik)}\Big|_{0}^{\infty} \\
&= \frac{n_0}{\alpha-ik} + \frac{n_0}{\alpha+ik} \\
&= \frac{2\alpha}{\alpha^2+k^2}
\end{aligned}$$

So the solution in k space becomes $Q(k,t) = \frac{2n_0\alpha}{\alpha^2+k^2}$. This then can be used to calculate the solution as

$$q(x,t) = \mathcal{F}^{-1}\Big(\frac{2n_0\alpha}{\alpha^2+k^2}e^{-k^2 t}\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2n_0\alpha}{\alpha^2+k^2}e^{ikx}e^{-k^2 t}dk = \frac{n_0\alpha}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-k^2 t}}{\alpha^2+k^2}dk$$

This gives the gemeral solution. ■