PHYS :502 Mathematical Physics II

Homework #5

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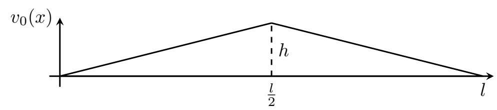
1. A string of length l is initially stretched straignt, its ends are fixed for all t. At $t = 0$, its points are given the velocity $v(x) = \left(\frac{\partial y}{\partial t}\right)_{t=0}$ s shown in the diagram. Determine the shape of string at time t, that is, find the displacement as a function of x and t in the form of a series.



Solution:
The motion of the string is guided by the wave equation which can be written as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

If we suppose the solution $y(x,t) = X(x)T(t)$ then substuting these and dividing thourgh by XT we obtain

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

The above solution is composed of two parts, each function of independent variables, the only way they can be equal is if they are equal to constant, let the constant that they are equal be k^2.

$$\frac{X''}{X} = k^2; \qquad \Rightarrow X = A\sin(kx) + B\cos(kx)$$
$$\frac{1}{c^2}\frac{T''}{T} = k^2; \qquad \Rightarrow T = D\sin(kct) + E\cos(kct)$$

So the solution to the differential equation becomes,

$$u(x,t) = [A\sin(kx) + B\cos(kx)][D\cos(kcx) + E\sin(kcx)]$$

But since the string is stationary at both ends. At $x = 0$ and $x = L$

$$0 = B\cos(kx)[D\cos(kct) + E\sin(kct)]$$

The only way it can be zero for all t is if $B = 0$. And also since the string has no displacement to begin with $u(x,0) = 0$. The only way this can happen similarly is if $D = 0$. The solution then becomes

$$u(x,t) = A\sin(kx)\sin(kct)$$

Also since $u(l,t) = 0$ for all t, the only way this can happen is if $k = \frac{n\pi}{l}$. Since we have different possible values of n for solution, the linear combination of all will be the most general solution

$$u(x,t) = \sum_n A_n \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{n\pi}{l}ct\right)$$

The velocity of the string at he begining is

$$u'(x,t) = \sum_n A_n \frac{n\pi c}{l} \sin\left(\frac{n\pi}{l}x\right) \cos\left(\frac{n\pi}{l}ct\right); \qquad \Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = u'(x,0) = \sum_n \frac{n\pi c}{l} \sin\left(\frac{n\pi}{l}x\right)$$

The coefficients A_n can be found by usual "Fourier Trick" as

$$A_n = \frac{2}{n\pi c} \int_0^l u'(x,0) \sin\left(\frac{n\pi}{L}x\right) dx$$

Since the given velocity function is two part function we obtain A_n as

$$\begin{aligned}
A_n &= \frac{2}{n\pi c} \left[\int_0^{l/2} u'(x,0) \sin\left(\frac{n\pi}{l}x\right) dx + \int_{l/2}^{l} u'(x,0) \sin\left(\frac{n\pi}{l}x\right) dx \right] \\
&= \frac{2}{n\pi c} \left[\int_0^{l/2} \frac{2h}{l} x \sin\left(\frac{n\pi}{L}x\right) dx + \int_{l/2}^{l} -\frac{2h}{l}(x-l) \sin\left(\frac{n\pi}{l}x\right) dx \right] \\
&= \frac{8hl}{\pi^3 c n^3} \sin\left(\frac{\pi n}{2}\right)
\end{aligned}$$

Substiting this back into the solution we have

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8hl}{\pi^3 c n^3} \sin\left(\frac{\pi n}{2}\right) \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{n\pi}{l}ct\right)$$

This gives the position of every point in the string as a function of time. ■

2. Consider the semi-infinite region $y > 0$. For $x > 0$, the surface $y = 0$ is maintained at a temperature $T_0 e^{-x/l}$, fo $x < 0$. The surface $y = 0$ is insulated, so that no head flows out or in. Find the equilibrium temperature at point $(-l, 0)$

Solution:

The general differential equation of the temperature diffusion is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\kappa} \frac{\partial}{\partial t} T$$

. But since at equilibrium the term $\frac{\partial}{\partial t} T = 0$. The general differential equation becomes

$$\frac{\partial^2}{\partial x^2} T(x,y) + \frac{\partial^2}{\partial y^2} T(x,y) = 0 \tag{1}$$

The temperature of the system $T(x,y)$ should go to zero as $y \to \infty$. Also since the surface $y = 0$ is insulated for $x < 0$. The heat flow at for $x < 0$ is $\frac{\partial}{\partial t} T(x,y)\big|_{0^-} = 0$. So by contunity of the function at $y = 0$, the rate of change of temperature with y at $y = 0^+$ should equal zero, so $\frac{\partial}{\partial y} T(x,0) = 0$. So the effective boundary condition becomes

$$T(x,0) = f(x) = \begin{cases} T_0 e^{-x/l} & (x > 0) \\ ? & (x < 0) \end{cases} \qquad \frac{\partial}{\partial y} T(x,y)\Big|_{y=0} = g(x) = \begin{cases} ? & (x > 0) \\ 0 & (x < 0) \end{cases}$$

Taking the fourier transform of (1) with respect to the variable x we get

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} T(x,y) e^{ikx} dx + \int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2} T(x,y) e^{ikx} dx; \qquad \Rightarrow -k^2 \tilde{T}(k,y) + \frac{\mathrm{d}^2}{\mathrm{d}y^2} \tilde{T}(k,y) = 0$$

Where $\tilde{T}(K, y)$ is the Fourier transform of $T(x, y)$ in variable x. Since this is a well known differential equation whose solution can be written as

$$\tilde{T}(k, y) = \Phi(k)e^{-ky}$$

Where $\Phi(k)$ is an unknown function to be determined by the boundary conditions. The required solution is the inverse Fourier transform is expression

$$T(x, y) = \mathcal{F}^{-1}(\tilde{T}(k, y)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-ky}\Phi(k)dk \tag{2}$$

Since we know the various parts at $y = 0$ substuting the anove function for $y = 0$ gives

$$T(x, 0) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\Phi(x)dx; \qquad \Rightarrow \Phi(x) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{3}$$

We will substitue k by $\sqrt{k^2 + \lambda^2}$ so that our solution will be in the limit $\lambda \to 0$.. Differentiating the function with respect to y and setting $y = 0$ we get

$$\frac{\partial}{\partial y}T(x, y) = g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial y}e^{-\sqrt{k^2+\lambda^2}y}\Phi(k)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\sqrt{k^2 + \lambda^2}e^{-\sqrt{k^2+\lambda^2}y}\Phi(k)dx$$

Setting $y = 0$ in above expression and taking fourier inverse transform of both sides gives

$$\sqrt{k^2 + \lambda^2}\Phi(x) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx \tag{4}$$

We can solve (3) and (4) with different parts of known $f(x)$ and $g(x)$. From (3) we get

$$\Phi(x) = \underbrace{\int_{-\infty}^{0} f(x)}_{\text{Unknown function}} e^{-ikx}dx + \int_{0}^{\infty} f(x)e^{-ikx}dx = \Phi_{-}(x) + \int_{0}^{\infty} T_0e^{-x/l}e^{-ikx}dx \tag{5}$$

$$= \Phi_{-}(x) + \frac{T_0}{i}\frac{1}{k - i/l} \tag{6}$$

Similarly solving (4) we get

$$\sqrt{k^2 + \lambda^2}\Phi(x) = \Psi_{+}(x) + 0 \tag{7}$$

Form (5) and (7) we get

$$\Psi_{+}(x) = \sqrt{k^2 + \lambda^2}\Phi_{-x} + \frac{T_0}{i}\frac{\sqrt{k^2 + \lambda^2}}{k - i/l}$$

Dividing by $\sqrt{k - i\lambda}$ on both sides we get

$$\sqrt{k + i\lambda}\Phi_{-}(k) - \frac{\Psi_{+}(k)}{\sqrt{k - i\lambda}} = \frac{T_0}{i}\frac{\sqrt{k + i\lambda}}{k - i/l} \tag{8}$$

Thie simplification this expression finally gives

$$T(x, y) = T_0\text{Re}\left[e^{-x-iy}\left(1 - \text{erf}\sqrt{-\frac{x + iy}{l}}\right)\right]$$

This is the solution for the temperature everywhere in the rod. At $(-l, 0)$ we get

$$T(-l, 0) = T_0e^l(1 - \text{erf}(1))$$

This gives the temperature at the required point. ■

3. (a) Deduce the relation $P'_{l+1} - P'_{l-1} = (2l+1)P_l$ and show that $\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}; \quad (l \geq 1)$

Solution:

The generating polynomial of the Legendre polynomial is

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_n P_n(x)t^n$$

Taking the partial derivative of both sides of the expression with respect to variable x we get

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1-2xt+t^2}}\right) = \frac{\partial}{\partial x}\sum_n P_n(x)t^n \qquad \Rightarrow \qquad \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_n P'_n(x)t^n$$

This can be pulated to get

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P'_n(x)t^n - t\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

$$\sum_{n=0}^{\infty} P'_n(x)t^n - \sum_{n=0}^{\infty} 2xP'_n(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^{n+2} = \sum_{n=0}^{\infty} P_n(x)t^{n+1}$$

Comparing the coefficient of t^{n+1} on both sides we get

$$P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) = P_n(x) \tag{9}$$

Again if we differentiate the generating function with respect to t and compare coefficients, we get

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{10}$$

If we differentiate the recurrence relation (10) we get

$$(2n+1)P_n(x) + (2n+1)xP'_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{11}$$

Also if we multiply (9) by $(2n+1)$ we get

$$(2n+1)P'_{n+1}(x) - 2(2n+1)xP'_n(x) + (2n+1)P'_{n-1}(x) = (2n+1)P_n(x) \tag{12}$$

If we subtract (11) from (12) we get

$$(2n+1)P_n = P'_{n+1}(x) - P'_{n-1}(x)$$

. This gives the required expression. The integral can be now written as

$$\int_0^1 P_l(x)dx = \int_0^1 \frac{P'_{l+1} - P'_{l-1}(x)}{2l+1}dx = \left[\frac{P_{l+1} - P_{l-1}(x)}{2l+1}\right]_0^1 = \left[\frac{P_{l+1}(1) - P_{l-1}(1) - P_{l+1}(0) + P_{l-1}(0)}{2l+1}\right]$$

Since $P_n(1) = 1$ for every n the expression simplifies to, and since there is $P_{l-1}(1)$ this will be 1 only if $l \geq 1$, which allows us to write,

$$\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}; \quad (l \geq 1)$$

This is te required integral of the Legendre polynomial in the given range. ■

(b) Show that $\int_0^l P_l(x)dx = \frac{P_{l-1}(0)}{l+1}; \qquad (l \geq 1).$

Solution:
From (a) we can write

$$\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}; \quad (l \geq 1)$$

We can the use (10) to evaluate $P_{l+1}(0)$ which gives

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}; \qquad P_{l+1}(0) = -\frac{l}{l+1}P_{l-1}(0)$$

Substuting this back we get

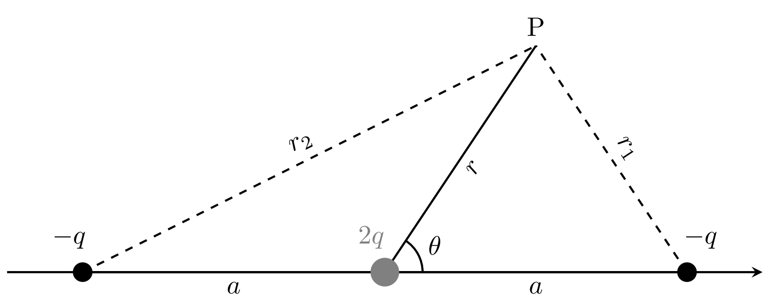
$$\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1} = \frac{1}{2n+1}\left[P_{l-1}(0) + \frac{l}{l+1}P_{l-1}\right] = \frac{1}{l+1}P_{l-1}(0)$$

Which gives the required expression for the integral. ■

4. A charge $+2q$ is situated at the origin and charges of $-q$ are situated at distances $\pm a$ from it along the polar axis. By relating it to the generating function for the Legendre polynomials, show that the electrostatic potential Φ at a point (r, θ, ϕ) withe $r > a$ is given by

$$\Phi(r, \theta, \phi) = \frac{2q}{4\pi\epsilon r}\sum_{s=1}^{\infty}\left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta).$$

Solution:
Let P be a a general point with coordinate (r, θ) in a particular plane. Since the potential only depends upon r and θ and there is no ϕ dependence, we can cancluate it for a plane polar case, which works for spherical polar as well.



Using the cosine law, the different quantities in the given diagram can be written as

$$r_1^2 = r^2 - 2ra\cos\theta + a^2; \qquad \Rightarrow \quad \left(\frac{r_1}{r}\right)^2 = 1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2$$

Since the generating function of legendre polynomial is

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

If we let $\frac{a}{r} = t$ and $\cos\theta = x$ we get

$$\frac{1}{r_1} = \frac{1}{r}\frac{1}{\sqrt{1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n \tag{13}$$

Similarly from r_2 from the diagram

$$r_2^2 = r^2 - 2ra\cos(\pi - \theta) + a^2; \qquad \Rightarrow \quad \left(\frac{r_1}{r}\right)^2 = 1 + 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2$$

Similarly from above expression we get

$$\frac{1}{r_2} = \frac{1}{r}\frac{1}{\sqrt{1 + 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(-\cos\theta)\left(\frac{a}{r}\right)^n \tag{14}$$

The potential any point P then becomes

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r} - \frac{q}{4\pi\epsilon_0 r_1} - \frac{q}{4\pi\epsilon_0 r_2} = \frac{q}{4\pi\epsilon_0 r}\left[2 - \frac{1}{r_1} - \frac{1}{r_2}\right]$$

Substuting r_1 and r_2 from (13) and (14) we get

$$V(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 r}\left[2 - \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty} P_n(-\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

Since $P_n(x) = (-1)^n P_n(-x)$ we get

$$V(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 r}\left[2 - \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty}(-1)^n P_n(\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

Which can be written as

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r}\left[1 - \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} P_n(\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

Since $P_0(x) = 1$ for all x we can simplify the expression

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r}\left[\sum_{n=1}^{\infty} \frac{1+(-1)^n}{2} P_n(\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

Since the expression $\frac{1+(-1)^n}{2} = 0$ for odd n wec can write

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r}\sum_{s=1}^{\infty} P_{2s}(\cos\theta)\left(\frac{a}{r}\right)^{2s}$$

Which is the required expression of the potential. ■