PHYS :502 Mathematical Physics II

Homework #4

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1. A slice of biological material of thickness L is placed into a solution of a radioactive isotope of constant concentration C_0, at time $t = 0$. For a later time t find the concentration of radioactive ions at a depth x inside one of its surfaces if the diffusion constant is κ.
Solution:
The diffusion equation with diffusion constant κ is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

Using the separation of variable technique for the solution the solution can be written as $u(x,t) = X(x)T(t)$ where X and T are pure functions of x and t respectively. Substituting this solution in the solution we get

$$\frac{X''}{X} = \frac{1}{\kappa}\frac{T'}{T} = -\lambda^2$$

The constant is chosen to be a negative number so that the exponential solution is finite at infinite time. The time part of solution is

$$\frac{T'}{T} = -\kappa\lambda^2; \quad \Rightarrow \int \frac{dT}{T} = \int -\kappa\lambda^2 dt; \quad \Rightarrow \ln T = -\kappa\lambda^2 t + K; \quad \Rightarrow T(t) = De^{-\kappa\lambda^2 t}$$

For the other part $\frac{X''}{X} = -\kappa\lambda^2$ has the solution of the form

$$A\sin\left(\frac{\lambda}{\sqrt{\kappa}}x\right) + B\cos\left(\frac{\lambda}{\sqrt{\kappa}}x\right)$$

The general solution then becomes

$$u(x,t) = \left[A\sin\left(\frac{\lambda}{\sqrt{\kappa}}x\right) + B\cos\left(\frac{\lambda}{\sqrt{\kappa}}x\right)\right]e^{-\lambda^2\kappa t}$$

After sufficient time has passed the concentration throughout the slab should be the concentration of isotopes around it. But the above solution goes to 0 at $t = \infty$. Since adding a constant to the above solution is still the solution to the diffusion equation. We can add a constant to make it satisfy this condition.

Since the concentration is constant at all times on either side of the slab, $u(0,t) = u(L,t) = C_0$ and so $X(0) = X(L) = 0$. So

$$X(0) = Be^{-\lambda^2\kappa t} = 0; \qquad \Rightarrow B = 0$$
$$X(L) = A\sin\left(\frac{\lambda}{\sqrt{\kappa}}L\right) = 0; \qquad \Rightarrow \frac{\lambda}{\sqrt{\kappa}}L = n\pi; \qquad \Rightarrow \lambda = \frac{n\pi\sqrt{\kappa}}{L}$$

Using these two facts we get our general solution to be

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} A_n \sin\Big(\frac{n\pi}{L}x\Big) e^{-\frac{n^2\pi^2\kappa}{L^2}t}$$

At $t = 0$ the concentration in the slab must be 0. So $u(x,0) = 0$

$$0 = u(x,0) = C_0 + \sum_{n=1}^{\infty} A_n \sin\Big(\frac{n\pi}{L}x\Big); \qquad \Rightarrow -C_0 = \sum_{n=1}^{\infty} A_n \sin\Big(\frac{n\pi}{L}x\Big)$$

Again the coefficients A_n can be calculated by using the fact that $\{\sin(nx)\}_n$ form an orthogonal set of function for integer set of n. Integrating the above expression by multiplying by $\sin\big(\frac{m\pi}{L}x\big)$ on both sides gives

$$\begin{aligned}
\int_0^L -C_0 \sin\Big(\frac{m\pi}{L}x\Big)dx &= \int_0^L \sum_n A_n \sin\Big(\frac{n\pi}{L}x\Big)\sin\Big(\frac{m\pi}{L}x\Big)dx \\
&= \sum_n A_n \frac{1}{L}\delta_{mn} = \frac{A_m}{L} \\
\Rightarrow \quad A_m = L\int_0^L -C_0 \sin\Big(\frac{m\pi}{L}x\Big)dx &= -LC_0\Big\{\frac{2}{m\pi}\frac{1+(-1)^m}{L}\Big\}
\end{aligned}$$

Using this the general solution becomes

$$u(x,t) = C_0 - \frac{2C_0}{\pi}\sum_m \frac{1+(-1)^m}{m}\sin\Big(\frac{m\pi}{L}x\Big)e^{-\frac{m^2\pi^2\kappa t}{L^2}}$$

This gives the concentration of radioactive isotope inside the slab at all times. ■

2. Determine the electrostatic potential in an infinite cyinder split lengthwise in four parts and charged as shown.

Solution:

Because the sides of cylindrical are conducting the potential is constant for $u(a,\phi,z)$ where a is the radius of cylinder. It follows that for all z, $u(\rho,\phi)$ is the same. So the potential satisfies plane polar form of laplaces equation which has the general solution

$$u(\rho,\phi) = (C_0 \ln\rho + D_0)\sum_n (A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n})$$

Since we expect finite solution at $\rho = 0$, $D_n = 0$ otherwise it $\rho^{-n} = \infty$ which won't satisfy boundary condition. By similar arguments $C_n = 0$ Also since at $\rho = a$ the solution is an odd function which causes $D_0 = 0$ and $A_n = 0$. The general solution that is left is

$$u(\rho,\phi) = \sum_n B_n\rho^n \sin n\phi$$

Again the coefficients B_n can be calculated by using the fact that $\{\sin n\phi\}_n$ form an orthogonal set of function for integer set of n. Integrating the above expression by multiplying by $\sin m\phi$ on both sides gives

$$\int_0^{2\pi} u(a,\phi)\sin m\phi d\phi = \int_0^{2\pi}\sum_n A_n a^n \sin n\phi \sin m\phi d\phi$$
$$= \sum_n A_n a^n \int_0^{2\pi} \sin n\phi \sin m\phi d\phi$$
$$= \sum_n A_n a^n \frac{2\pi}{2}\delta_{mn} = A_m a^m \pi$$
$$\Rightarrow \quad A_m = \frac{1}{\pi a^m}\int_0^{2\pi} u(a,\phi)\sin m\phi d\phi$$

Since in the given problem $u(a,\phi)$ has different values for different ϕ we get

$$A_m = \frac{1}{\pi a^m}\left\{\int_0^{\pi/2} V\sin m\phi d\phi - \int_{\pi/2}^{\pi} V\sin m\phi d\phi + \int_{-\pi}^{-\pi/2} V\sin m\phi d\phi - \int_{-\pi/2}^{2\pi} V\sin m\phi d\phi\right\}$$
$$= \frac{V}{\pi a^m}\left\{-\frac{1}{m}\cos\left(\frac{\pi m}{2}\right) + \frac{1}{m} - \frac{(-1)^m}{m} + \frac{1}{m}\cos\left(\frac{\pi m}{2}\right)\frac{(-1)^m}{m} - \frac{1}{m}\cos\left(\frac{\pi m}{2}\right)\frac{1}{m}\cos\left(\frac{\pi m}{2}\right) - \frac{1}{m}\right\}$$
$$= \frac{V}{\pi a^m}\left\{\frac{2(-1)^m}{m} - \frac{4}{m}\cos\left(\frac{\pi m}{2}\right) + \frac{2}{m}\right\}$$
$$= \frac{V}{\pi a^m}\left\{\frac{1}{m}\left(-(-1)^{\frac{m}{2}}((-1)^m+1) - (-1)^m + 1\right)\right\}$$

So the final solution becomes

$$u(\rho,\phi) = \frac{V}{m\pi}\left\{1-(-1)^{\frac{m}{2}}((-1)^m) - (-1)^m + 1\right\}\left(\frac{\rho}{a}\right)^m \sin(m\phi)$$

This gives the potential everywhere inside the cylinder. ■

3. A heat-conducting cylindrical rod of length L is thermally isnulated over its lateral surface and its ends are kept at zero temperature. the initial temperature of the rod is $u(x) = u_0$. using the diffusion equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

and the boundary conditions $u(0,t) = u(L,t) = 0$ and the initial condition $u(x,0) = u_0$, obtain the solution $u(x,t)$ of the above equation.
Solution:
The general solution to the diffusion equation is

$$u(x,t) = (A\sin(\lambda x) + B\cos(\lambda x))e^{-\lambda^2 a^2 t}$$

Given initial condition $u(0,t) = 0$

$$u(0,t) = e^{-\lambda^2 a^2 t}(B\cos(\lambda x)) = 0$$

Since function has to be 0 at all times the only way this can happen for all t is $B = 0$ Also the other boundary condition is $u(L,t) = 0$ gives

$$u(L,t) = e^{-\lambda^2 a^2 t}A\sin(\lambda L) = 0$$

Since $A = 0$ will give us the trivial solution 0 the only way this function can go to zero at all time is $\sin(\lambda L) = 0$ which implies

$$\sin(\lambda L) = 0; \quad \Rightarrow \lambda L = n\pi; \qquad \Rightarrow \lambda = \frac{n\pi}{L}; \quad (n \geq 1)$$

Since the solution can be linear combination of all n so the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda^2 a^2 t} \sin\Big(\frac{n\pi}{L}x\Big)$$

But since the initial condition is that the temperature of the rod is u_0 to begin with. The above solution clearly goes to zero at $t = 0$ and $x = 0$. Adding a constant to a solution of differential equation is still a valid solution, to satisfy this condition we can add a constant u_0. The valid general solution then becomes

$$u(x,t) = u_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda^2 a^2 t} \sin\Big(\frac{n\pi}{\lambda}x\Big)$$

At $t = 0$ the the solution reduces to

$$u(x,0) = u_0 + \sum_{n=1}^{\infty} A_n \sin\Big(\frac{n\pi}{L}x\Big); \qquad \Rightarrow \sum_{n=1}^{\infty} A_n \sin\Big(\frac{n\pi}{L}x\Big) = -u_0$$

Sine $\sin(nx)$ forms an orthogonal set of function for integer set of n. We can find A_n by integrating above expression multiplied with $\sin mx$

$$\begin{aligned}
\int_0^l -u_0 \sin\Big(\frac{m\pi}{L}x\Big)dx &= \int_0^l \sum_n A_n \sin\Big(\frac{n\pi}{L}x\Big)\sin\Big(\frac{m\pi}{L}\Big)dx \\
&= \sum_n A_n \int_0^l \sin\Big(\frac{n\pi}{L}\Big)\sin\Big(\frac{m\pi}{L}x\Big)dx \\
&= \sum_n A_n \frac{l}{2}\delta_{nm} = \frac{l}{2}A_m \\
\Rightarrow \qquad A_m &= -\frac{2u_0}{l}\int_0^l \sin\Big(\frac{m\pi}{L}x\Big)dx = -\frac{2u_0}{l}\Big(l\frac{1-(-1)^m}{m}\Big)
\end{aligned}$$

Using this in the solution we get the final solution as

$$u(x,t) = u_0 - 2u_0 \sum_{m=1}^{\infty} \Big(\frac{1-(-1)^m}{m}\Big) \sin\Big(\frac{m\pi}{L}\Big) e^{-\lambda^2 a^2 t}$$

This gives the temperature as a function of position and time in the given cylindrical body. ■

4. Consider the semi-infinite heat conducting medium defined by the region $x \geq 0$, and arbitrary y and z. Let it be initially at at 0 temperature and let its surface $x = 0$, have prescribed variation of temperature $u(0,t) = f(t)$ for $(t \geq 0)$. Show that the solution of the above diffusion equation can be written as

$$u(x,t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{3/2}} f(\tau)d\tau$$

Solution:
Since the temperature conduction of a material satisfies the diffusion equation, the diffusion equation can be written as.

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Since the parameters of this problems are $t \to \{0, \infty\}$ and $x \to \{0, \infty\}$, we can take the laplace transform of the equation with respect to the variable t which results in

$$\int_0^\infty a^2 \frac{\partial^2}{\partial x^2} u(x,t) e^{-st} dt = \int_0^\infty \frac{\partial}{\partial t} u(x,t) e^{-st} dt$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_0^\infty u(x,t) e^{-st} dt = \frac{1}{a^2} \int_0^\infty \frac{\partial}{\partial t} u(x,t) e^{-st} dt$$

Assuming $u(x,t) = g(t)$, the RHS of above expression is the laplace transform of derivative of $g(t)$ which is $sG(t) - g(0)$ which can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} U(x,s) = \frac{1}{a^2}(sU(x,s) - u(x,0))$$

The term $u(x,0)$ is the initial temperature of the material body under construction, since the body is initially at 0 temperature $u(x,0) = 0$, using this and rearranging gives

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} U(x,s) - \frac{s}{a^2} U(x,s) = 0$$

This is a very well known second order Ordinary Differential equation whose solution is of the form

$$U(x,s) = Ae^{-x\sqrt{s}/a} + Be^{x\sqrt{s}/a}$$

But since the material body is infinitely long in $x \geq 0$ the solution is finite at $x = \infty$ which implies that $B = 0$. Also at the near end of the material $x = 0$ the temperature $u(0,t) = f(t)$ is given. The laplace transform of which is $U(0,s) = F(s)$. So

$$U(0,s) = Ae^0; \qquad \Rightarrow A = U(0,s) = F(s)$$

This reduces the solution in the form

$$U(x,s) = F(s) e^{-x\sqrt{s}/a}$$

At this point the solution $u(x,t)$ is the inverse laplace transform of $U(x,s)$. If the expression is taken as product of $F(s)$ and $e^{-x\sqrt{s}/a}$ the solution is the convolution of inverses of these.

Looking at the result we expect, the inverse laplace transfrom must tbe

$$\mathcal{L}^{-1}\left\{e^{-x\sqrt{s}/a}\right\} = \frac{xe^{-\frac{x^2}{4a^2 t}}}{2\sqrt{\pi} a t^{\frac{3}{2}}}$$

I checked this in sympy and got the following

So the inverse laplace transform of $U(x,s)$ is

$$u(x,t) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}\left\{e^{-x\sqrt{s}/a}\right\} = \int_0^\infty \frac{x}{2\sqrt{\pi} a} \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{3/2}} f(\tau) d\tau$$

In [1]:
```
import sympy as smp
from sympy.integrals import transforms as strn
smp.init_printing();
```

In [2]:
```
x,t = smp.symbols('x,t',real=True,positive=True)
s = smp.symbols('s',complex=True)
a,k = smp.symbols('a,kappa',constant=True)
```

In [3]:
```
#this is the expected laplace transform of the function we need
fx = x/(2*a*smp.sqrt(smp.pi))*smp.exp(-x**2/(4*a**2*t))/t**smp.Rational(3,2) ; fx
```

Out[3]:

$$\frac{xe^{-\frac{x^2}{4a^2t}}}{2\sqrt{\pi}at^{\frac{3}{2}}}$$

In [4]:
```
#calculates the laplace transform of above function
strn.laplace_transform(fx,t,s)
```

Out[4]:

$$\left(e^{-\frac{\sqrt{s}x}{a}}, \quad -\infty, \quad 0 < \Re(s) \wedge \left|\operatorname{periodic}_{\operatorname{argument}}\left(\operatorname{polar\_lift}^2(a), \infty\right)\right| < \frac{\pi}{2}\right)$$

Since the integration is with respect to τ the variable x is consant for integration which leads to

$$u(x,t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{3/2}} f(\tau)d\tau$$

Which is the required solution of the heat equation. ■