PHYS :502 Mathematical Physics II

Homework #3

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1. A cube made of material whose conductivity is k has its six faces the planes $x = \pm a, y = \pm a$ and $z = \pm a$, and contains no internal heat sources. Verify that the temperature distribution

$$u(x,y,z) = A\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{a}\right)\exp\left(-\frac{2k\pi^2 t}{a^2}\right)$$

obeys the appropirate diffusion equation. Across which faces is there heat flow? WHat is the direction and the rate of heat flow at the point $\left(\frac{3a}{4}, \frac{a}{4}, a\right)$ at time $t = a^2/(\kappa\pi^2)$?

Solution:

Since the expression is the product of sinusoids and exponentials, the derivatives are easy to calculate and are by inspection

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\pi^2}{a^2}u; \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\pi^2}{a^2}u; \qquad \frac{\partial u}{\partial t} = -2\frac{\kappa\pi^2}{a^2}$$

Checking this on the diffusion equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 2\frac{\pi^2}{a^2} = \frac{1}{\kappa} - 2\frac{\pi^2}{a^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

clearly satisfies it, Showing this function obeys the temperature diffucsion equation. The direction of heat flow is given by the gradient of function. At $t = \frac{a^2}{\kappa\pi^2}; u = A\cos(x\pi/a)\sin(z\pi/a)e^{-2}$

$$\begin{aligned}\nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial z}\hat{k} &= A\frac{\pi e^{-2}}{a}(-\sin(x\pi/a)\sin(z\pi/a)\hat{i} + \cos(x\pi/a)\cos(z\pi/a)\hat{k}) \\ &= A\frac{e^{-2}\pi}{a}(-\sin(\pi/4)\sin(\pi)\hat{i} + \cos(\pi/4)\cos(\pi)\hat{k}) \\ &= A\frac{e^{-2}\pi}{a}\left(-\frac{1}{\sqrt{2}}\hat{k}\right)\end{aligned}$$

So the rate of heat flow is $\dfrac{Ae^{-2}\pi}{a\sqrt{2}}$ in the direction of $-\hat{k}$ ■

2. Schrodinger's equation for an non=reativistic particle ina constant potential region can be taken as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = i\hbar\frac{\partial u}{\partial t}$$

   (a) Find a solution, separable in the four independent variables, that can be written in the form of a plane wave

$$\psi(x,y,z,t) = A\exp(i(\mathbf{k}.\mathbf{r} - \omega t))$$

Using the relationships associated with de Broglie ($\mathbf{p} = \hbar\mathbf{k}$) and Einstein ($E = \hbar\omega$), show that the separation constants must be such that

$$p_x^2 + p_y^2 + p_z^2 = 2mE$$

Solution:
Lets assume the solution $u(x, y, z, t) = XYZT$ where X is purely function of x only and so on with T being pure function of t. Substuting this product in the given PDE we get

$$-\frac{\hbar^2}{2m}(X''YZT + XY''ZT + XYZ''T) = i\hbar XYZT'$$

Where X'' and so on are total second derivative of their only parameters, x and so on. Dividing thorough by the product $XYZT$ we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = i\hbar\frac{T'}{T}$$

Since we assumed that each X, Y, Z, and T are independent of each other the only way the function of independent variables can be equal is if they are each equal to a constant. Let the constant that each side are equal be E. So we get.

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = i\hbar\frac{T'}{T} = E \quad \text{(Separation Constant)}$$

Solving the ordinary differential equan in t we get

$$\frac{T'}{T} = -i\frac{E}{\hbar}; \qquad \Rightarrow \frac{dT}{T} = -i\frac{E}{\hbar}dt; \quad \Rightarrow \ln(T) = -i\frac{E}{\hbar}t; \qquad \Rightarrow T = T_0e^{-i\omega t}; \text{Where } \omega = \frac{E}{\hbar}$$

Also the LHS must equal same constant so

$$\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = -\frac{2mE}{\hbar^2}$$

The LHS of this expression is sum of three independent functions and the RHS is a constant void of any variables under considerations. The only way that can happen is if each independent function is a constant

$$\frac{X''}{X} = -k_x^2; \qquad \frac{Y''}{Y} = -k_y^2; \qquad \frac{Z''}{Z} = -k_z^2$$

Substuting these back in the differential equation imply that they are related by the expression $-k_x^2 - k_y^2 - k_z^2 = -\frac{2mE}{\hbar^2}$. If we write $p_x = \hbar k_x, p_y = \hbar k_y$, and $p_z = \hbar k_z$. Then we get

$$p_x^2 + p_y^2 + p_z^2 = 2mE \tag{1}$$

Each ODE in X, Y and Z are well known Harmonic oscillator differential equations and the solution of each are

$$X = X_0e^{-ik_xx}; \qquad Y = Y_0e^{-ik_yy}; \qquad Z = Z_0e^{-ik_zz} \tag{2}$$

Where each of X_0, Y_0 and Z_0 are constants. Combining all these in our final solution we get

$$\begin{aligned} u(x, y, z, t) = XYZT &= X_0e^{-ik_xx} \cdot Y_0e^{-ik_yy} \cdot Z_0e^{-ik_zz} \cdot T_0e^{-i\omega t} \\ &= X_0Y_0Z_0T_0e^{-ik_xx - ik_yy - ik_zz - i\omega t} \end{aligned}$$

If we write $A = X_0Y_0Z_0T_0$, $\quad \mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_x\hat{\mathbf{z}}$ and $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + x\hat{\mathbf{z}}$ then the solution takes the form

$$u(x, y, z, t) = Ae^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{3}$$

Which is the required solution of the given Schrodinger's equation. ■

(b) Obtain a different separable solution describing a particle confined to a box of side a (ψ must vanish at the walls of the box). Show that the energy of the particle can only take quantized values

$$E = \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

where n_x, n_y, n_z are integers.
Solution:
If the solution vanish at the wall of box then each solution given by (**??**) should vanish at the wall. So this iimplies

$$\begin{aligned} 0 &= X_0e^{-ik_xa} & 0 &= Y_0e^{-ik_ya} & 0 &= Z_0e^{-ik_za} \\ \Rightarrow k_xa &= \pi n_x & k_ya &= \pi n_y & \Rightarrow k_za &= \pi n_z \\ \Rightarrow k_x &= \frac{\pi n_x}{a} & k_y &= \frac{\pi n_y}{a} & \Rightarrow k_z &= \frac{\pi n_z}{a} \end{aligned}$$

Substuting these values in (**??**) we get

$$-\left(\frac{n_x\pi}{a}\right)^2 - \left(\frac{n_y\pi}{a}\right)^2 - \left(\frac{n_z\pi}{a}\right)^2 = -\frac{2mE}{\hbar^2}; \qquad \Rightarrow \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right) = E$$

Which is the required solution ■

3. Consider possible solutions of Laplaec's equation inside a circular domain as follows

(a) Find the sollution in plane polar corrdinates ρ, ϕ that takes the value $+1$ for $0 < \phi < \pi$ and the value -1 for $-\pi < \phi < 0$ where $\rho = a$.
Solution:
The general solution for the Laplace's equation in plane polar coordinate system, where the solution is finite at $\rho = 0$ is

$$u(\rho, \phi) = D + \sum_n (C_n\rho^n)(A_n \cos n\phi + B_n \sin n\phi)$$

Since the given boundary condition is an odd function of *phi*, the even function term in the above general solution must vanish so, $D = 0$ and $A_n = 0$. The remaining general solution is

$$u(\rho, \phi) = \sum_n \rho^n(B_n \sin n\phi)$$

Where C_n is absorbed inside of B_n ■

(b) For a point (x, y) on or inside the circle $x^2 + y^2 = a^2$, identify the angles α and β defined by

$$\alpha = \text{atan}\left(\frac{y}{a+x}\right); \qquad \text{and} \qquad \beta = \text{atan}\left(\frac{y}{a-x}\right)$$

Show that $u(x, y) = (2/\pi)(\alpha + \beta)$ is a solution of Laplace's equation that satisfies the boundary conditions given in (**??**).

Solution:
Using the trigonometric identity of inverse tangents we get

$$u(x,y) = \frac{2}{\pi}(\alpha+\beta) = \frac{2}{\pi}\left(\text{atan}\left(\frac{y}{a+x}\right) + \text{atan}\left(\frac{y}{a-x}\right)\right) = \frac{2}{\pi}\,\text{atan}\left(\frac{2y}{a^2-x^2-y^2}\right)$$

To verify that $u(x,y)$ satisfies the Laplace's equation we have to show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Calculating this expression

$$\frac{\partial u}{\partial x} = \frac{2y\left(-(a-x)^2+(a+x)^2\right)}{\pi\left(y^2+(a-x)^2\right)\left(y^2+(a+x)^2\right)}; \quad \frac{\partial u}{\partial y} = \frac{2\left((a-x)\left(y^2+(a+x)^2\right)+(a+x)\left(y^2+(a-x)^2\right)\right)}{\pi\left(y^2+(a-x)^2\right)\left(y^2+(a+x)^2\right)}$$

Similarly the second partial derivatives of each is

$$\frac{\partial^2 u}{\partial x^2} = \frac{4ay}{\pi\left(y^2+(a+x)^2\right)^2} + \frac{4ay}{\pi\left(y^2+(a-x)^2\right)^2} + \frac{4xy}{\pi\left(y^2+(a+x)^2\right)^2} - \frac{4xy}{\pi\left(y^2+(a-x)^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{8ay\left(-a^4-2a^2x^2-2a^2y^2+3x^4+2x^2y^2-y^4\right)}{\pi\left(a^2-2ax+x^2+y^2\right)^2\left(a^2+2ax+x^2+y^2\right)^2}$$

On adding $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ we find that it is identically zero. So it satisfies the laplace's equation. On at the boundary $a^2 = x^2+y^2$ and inside the boundary $a^2 > x^2+y^2$ so $a^2 \geq x^2+y^2$. On the boundary

$$u(x,y) = \frac{2}{\pi}\,\text{atan}\left(\frac{2y}{a^2-x^2-y^2}\right) = \frac{2}{\pi}\text{sgn}(2y)\frac{\pi}{2} = \text{sgn}(y)$$

Where $\text{sgn}(x)$ is the sign function. But on boundary $y = a\sin\phi$ where a is the radius and ϕ is the azimuthal angle. The function $\sin\phi$ is positive for $0<\phi<\pi$ and negative for $-\pi<\phi<0$, so

$$u(x,y) = \text{sgn}(y) = \text{sgn}(\sin\phi) = \begin{cases} 1 & 0<\phi<\pi \\ -1 & -\pi<\phi<0 \end{cases}$$

Thus the function satisfies Laplace's equation and also the boundary condition. ■

(c) Deduce a Fourier series expansion for the function

$$\text{atan}\left(\frac{\sin\phi}{1+\cos\phi}\right) + \text{atan}\left(\frac{\sin\phi}{1-\cos\phi}\right)$$

Solution:
Again by trigonometric identity

$$f(\phi) = \text{atan}\left(\frac{\sin\phi}{1+\cos\phi}\right) + \text{atan}\left(\frac{\sin\phi}{1-\cos\phi}\right) = \text{atan}\left(\frac{2\sin\phi}{1-\sin^2\phi-\cos^2\phi}\right) = \frac{\pi}{2}\text{sgn}(\sin\phi) = \begin{cases} \pi/2 & 0<\phi<\pi \\ -\pi/2 & -\pi<\phi<0 \end{cases}$$

Let the fourier series of this function $f(\phi)$ be

$$f(\phi) = \frac{a_0}{2} + \sum_n a_n\cos n\phi + b_n\sin n\phi$$

This is a well known periodic square wave function. It is an odd function so $a_n = 0$ whose fourier series is given by

$$a_n = 0; \text{ and } b_n = \frac{\pi}{n}(1-(-1)^n)$$

So the required fouerier series of the function is

$$f(\phi) = \text{atan}\left(\frac{\sin\phi}{1+\cos\phi}\right) + \text{atan}\left(\frac{\sin\phi}{1-\cos\phi}\right) = \sum_{n=1}^{\infty} \frac{\pi}{n}(1-(-1)^n)\sin n\phi$$

This is the required fourier series of the function. ■

4. A conducting spherical shell of radius a cut round its equator and the two halves connected to voltages $+V$ and $-V$. Show that an expression for the potential at the point (r, θ, ϕ) anywhere inside the two hemispheres is

$$u(r,\theta,\phi) = V\sum_{n=0}^{\infty} \frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(n+1)!}\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta)$$

Solution:
For the spherical split spherical shell maintained at two differential potentials, let the potential everywhere inside the spherical shell be v. Since we know electric field is given by $\mathbf{E} = \nabla v$ and since for Electric field $\nabla \cdot \mathbf{E} = 0$. We get $\nabla \cdot \mathbf{E} = \nabla \cdot \nabla v = \nabla^2 v = 0$. So the potential satisfies the Laplace's equation. If we suppose v as a function of r, θ, ϕ in spherical coordinate system, then the solution to Laplace's equation in spherical coordinate system is given by

$$v(r,\theta,\phi) = \sum_{l,m}^{\infty}(Ar^l + Br^{-(l+1)})(C\cos m\phi + D\sin m\phi)(EP_l^m(\cos\theta) + FQ_l^m(\cos\theta))$$

Where $Q_l^m(x)$ and $P_l^m(x)$ are solution to the associated Legendre's equations. And all other constants are determined by boundary condition.

Since we have finite potential at at the center of sphere $r = 0$, the coefficient $B = 0$. Also since we have spherical symmetry and the potential is single valued function $m = 0$. Also we have finite potential at poles of sphere wchich correspond to $\theta = \{0, \pi\}$ and $Q_l^m(1)$ diverges, we have $F = 0$. Also $P_l^0(x) = P_l(x)$ where $P_l(x)$ are legendre polynomials. Owing to these boundary conditions the most general solution is

$$v(r,\theta,\phi) = \sum_l A_l r^l P_l(\cos\theta) \tag{4}$$

Since there is no ϕ dependence, let the potential at surface be denoted by v_a which is clearly jut function of θ.

$$v_a(\theta) = v(a,\theta,\phi) = \sum_l A_l a^l P_l(\cos\theta)$$

If we multiply both sides by $P_k(\cos\theta)$ and and integrate with respect to $d(\cos\theta)$ from 0 to 1 using the fact that Legendre's polynomials are orthogonal, $\int P_k P_l = \delta_{kl}$ we get.

$$\begin{aligned}\int_0^1 v_a(\theta)P_k(\cos\theta)d(cos\theta) &= \int_0^1\left(\sum_l A_l a^l P_l(\cos\theta)P_k(\cos\theta)d(\cos\theta)\right)\\ &= \sum_l\left(\int_0^1 A_l a^l P_l(\cos\theta)P_k(\cos\theta)d(\cos\theta)\right)\\ &= \sum_l A_l a^l \delta_{lk} = A_k a^k\end{aligned}$$

So the coefficient A_k is given by

$$A_k = \frac{1}{a^k}\int_0^1 v_a(\theta)P_k(\cos\theta)d(\cos\theta) \tag{5}$$

The recurrance relation of Legendre polynomials can be used to evaluate the integrals as

$$(2n+1)P_n = P'_{n+1}(x) - P'_{n-1}(x) \tag{6}$$

Integrating (**??**) we get,

$$\int P_n = \frac{1}{2n+1}(P_{n+1}(x) - P_{n-x}(x)) + K$$

Since Potential can have any arbitrary reference we can choose the integration constant to be $K = 0$. Using this fact in (**??**) we get

$$A_k = \frac{1}{a^k(2n+1)} \tag{7}$$

As given in the problen on the upper hemisphere the potential is $+V$ and on the lowe hemisphere the potential is $-V$, It can be mathematically represented as

$$v_a(\theta) = \begin{cases} V & \text{if } 0 < \theta < \frac{\pi}{2} \\ -V & \text{if } \frac{\pi}{2} < \theta < \pi \end{cases}$$

Substuting this in (**??**) we get and writing $x = \cos\theta$

$$\begin{aligned} A_k &= \frac{1}{a^k}\int_0^1 VP_k(x)dx \\ &= \frac{V}{a^k}\frac{1}{2k+1}\Big([P_{k+1}(x) - P_{k-1}(x)]_0^1\Big) \\ &= \frac{V}{a^k}\frac{1}{2k+1}(P_{k+1}(1) - P_{k-1}(1) - P_{k+1}(0) + P_{k-1}(0)) \\ &= \frac{V}{a^k}\frac{1}{2k+1}(P_{k-1}(0) - P_{k+1}(0)) \end{aligned}$$

Since

$$P_n(0) = \begin{cases} \dfrac{(-1)^n(2n)!}{2^{2n}n!^2}, & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

For evn value of k, both $k-1$ and $k+1$ are odd and hence $P_{k-1}(0) = 0$ and $P_{k-1}(0) = 0$. Foe even k,

$$A_k = \frac{V}{a^k}\frac{1}{2k+1}(0 - 0 - 1 + 1) = 0$$

But for odd value of k, $k+1$ and $k-1$ are even, hence both $P_{k-1}(1) = P_{k+1}(1) = 1$ and writing $k = 2n+1$

$$\begin{aligned} A_k &= \frac{V}{a^k}\frac{1}{4n+3}\left(\frac{(-1)^{2n}(2(2n)!}{2^{2(2n)}(2n)!^2} - \frac{(-1)^{2(n+1)}(2(2(n+1))!}{2^{2(2(n+1))}(2(n+1))!^2}\right) \\ &= \frac{(4n!)}{den} \qquad = \frac{V}{a^k}\frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(n+1)!} \end{aligned}$$

Using this coefficient in (**??**) we get

$$v(r,\theta,\phi) = V\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(n+1)!}\left(\frac{r}{a}\right)^{2k+1}P_{2n+1}(\cos\theta)$$

Which is the required potential function inside the spherical region. ■