PHYS :502 Mathematical Physics II

Homework #2

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1. Solve the differential equation $y'' - 4y' + y = 0; y(0) = 0; \quad y'(0) = 0$ using the laplace transformation.
   Solution:
   Let $Y(S)$ be the laplace transformation of $y(x)$. Taking the laplace transformation of given differential equation

$$\mathcal{L}\{y'' - 4y' + y\} = \mathcal{L}\{0\}$$
$$s^2Y(s) - y(0) - sy'(0) - 4sY(s) + 4y(0) + Y(s) = 0$$

   Substuting the given initial conditions $y'(0) = 0$; and $y(0) = 0$ gives

$$(s^2 - 4s + 1)Y(s) = 0; \qquad \Rightarrow Y(s) = \frac{1}{s^2 - 4s + 1} = \frac{1}{(S-2)^2 - 3} = \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(S-2)^2 - \sqrt{3}^2}$$

   The laplace laplace transform is in the form $\frac{a}{(s-m)^2 - a^2}$ and the laplace inverse of this expression is

$$y(s) = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(S-2)^2 - \sqrt{3}^2}\right\} = \frac{1}{\sqrt{3}} e^{2t} \sinh(\sqrt{3}t)$$

   This is the required solution for the differential equation. ■

2. Using the convolution theorem establish the following result:

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} = \frac{1}{2}\left(t\cos(at) + \frac{1}{a}\sin(at)\right)$$

   Solution:
   The given expression can be written as

$$\frac{s^2}{(s^2 + a^2)^2} = \frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2}$$

   It is easily recognized that each part is the Laplace transform of $cos(at)$. So the inverse Laplace transform by convolution theorem is the convolution of $\cos(at)$ with itself

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\cdot\right\} = \cos(at) * \cos(at) &= \int_0^t \cos(ax)\cos(at-ax)dx \\
&= \int_0^t \cos(ax)(\cos(at)\cos(ax) + \sin(at)\sin(ax))dx \\
&= \cos(at)\int_0^t \cos^2(ax)dx + \frac{1}{2}\sin(at)\int_0^t \sin(2x)dx \\
&= \cos(at)\left[\frac{2ax+\sin(2ax)}{4a}\right]_0^t + \frac{1}{2}\sin(at)\left[-\frac{\cos(2ax)}{2a}\right]_0^t \\
&= \frac{1}{2}t\cos(at) + \frac{2}{4a}\sin(at)\cos^2(at) - \frac{1}{2}\sin(at)\cdot\frac{2\cos^2(at)}{2a} + \frac{1}{4a}\sin(at) + \frac{1}{2}\frac{\sin(at)}{2a} \\
&= \frac{1}{2}\left(t\cos(at) + \frac{1}{a}\sin(at)\right)
\end{aligned}
$$

This is the required inverse fourier transform for the given expression. ■

3. Show that $\mathcal{L}\left\{Ci(t)\right\} = -\frac{1}{2s}\ln(1+s^2)$ where $C(i) = -\int_t^\infty \frac{\cos(u)}{u}du$ (The cosine integral).
Solution:
We know the differential under integral is

$$
\frac{d}{dt}\int\limits_{u(t)}^{v(t)} f(t,u)du = f(t,v(t))\frac{dv(t)}{dt} - f(t,u(t))\frac{du(t)}{dt} + \int\limits_{u(t)}^{v(t)} \frac{\partial f(u,t)}{\partial t}du
$$

Considering $f(t,u) = \frac{\cos(u)}{u}$, $v(t) = R(\text{as } R- \to \infty)$ and $u(t) = t$ we get

$$
\begin{aligned}
\frac{dCi(t)}{dt} &= \frac{\cos(R)}{R}\overbrace{\frac{dR}{dt}}^{0} - \frac{cos(t)}{t}\frac{dt}{dt} + \lim_{R\to\infty}\int\limits_t^R \overbrace{\frac{\partial}{\partial t}\left(\frac{\cos(u)}{u}\right)}^{0}du = -\frac{\cos(t)}{t} \\
\Rightarrow \qquad tCi'(t) &= \cos(t)
\end{aligned}
$$

Taking the laplace transform of both sides and writing $CI(s) \equiv \mathcal{L}\left\{Ci(t)\right\}$ we get

$$
\begin{aligned}
\mathcal{L}\left\{tCi'(t)\right\} &= \mathcal{L}\left\{\cos(t)\right\} \\
\Rightarrow \qquad -\frac{d}{ds}\mathcal{L}\left\{Ci'(t)\right\} &= \frac{s}{s^2+1} \\
-\frac{d}{ds}\left(sCI(S) - Ci(0)\right) &= \frac{s}{s^2+1} \\
-\frac{d(sCI(s))}{ds} &= \frac{s}{s^2+1}
\end{aligned}
$$

This expression is an ordinary differential equation which can be solved as

$$
\begin{aligned}
-\int d(sCI(s)) &= \int \frac{ds}{s^2+1} \\
\Rightarrow \qquad -sCI(s) &= \frac{1}{2}\ln(s^2+1) \\
CI(s) &= -\frac{1}{2s}\ln(s^2+1)
\end{aligned}
$$

This is the required Laplace transform of $Ci(s) \equiv CI(s) = -\frac{1}{2s}\ln(s^2+1)$. ■

4. By performing the rational fraction decomposition, establish the following results:

   (a) $\mathcal{L}^{-1}\left\{\frac{s+1}{s(s^2+1)}\right\} = 1 + \sin(t) - \cos(t)$

   Solution:

   The partial fraction of

$$\frac{s+1}{s(s^2+1)} = \frac{1}{s} - \frac{s-1}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

   Now the inver se laplace transform is

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= 1 - \cos(t) + \sin(t)\end{aligned}$$

   Which is the required inverse laplace transform ■

   (b) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+s+1)}\right\}$

   Solution:

   The partial fraction of this expression is

$$\frac{s+1}{s^2(s^2+s+1)} = -\frac{1}{s^2+s+1} + \frac{1}{s^2} = \frac{1}{s^2} - \frac{1}{(s+1/2)+1-1/4} = \frac{1}{s^2} - \frac{1}{(s+1/2)+(\sqrt{3}/2)^2}$$

   The inverse laplace transform is

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+s+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)+(\sqrt{3}/2)^2}\right\} \\ &= t - \frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\end{aligned}$$

   Which is the required inverse transform. ■