PHYS :502 Mathematical Physics II

Homework #1

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1. Use the general definition and properties of Fourier transforms to show the following

(a) If $f(x)$ is periodic with period a then $\tilde{f}(k) = 0$, unless $ka = 2\pi n$ for integer n.
Solution:
We know by definition of fourier transform

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-ikt}dt = \tilde{f}(k) \qquad \text{(Defition)}$$

$$\mathcal{F}(f(t-a)) = \int_{-\infty}^{\infty} f(t-a)e^{-ikt}dt = e^{-ika}\tilde{f}(k) \qquad \text{(Shifting property)}$$

Since the function is periodic $f(t) = f(t-a)$ and hence $\mathcal{F}(f(t)) = \mathcal{F}(f(t-a))$. So,

$$\tilde{f}(k) = e^{-ika}\tilde{f}(k); \quad \Rightarrow (e^{-ika} - 1)\tilde{f}(\omega) = 0;$$

Either $f(k) = 0$ Or $e^{-ika} = 1$; $\quad \Rightarrow ka = 2\pi n$. Which completes the proof. ∎

(b) The Fourier transform of $tf(t)$ is $d\tilde{f}(\omega)/d\omega$.
Solution:

$$\frac{d}{d\omega}\left(\tilde{f}(\omega)\right) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{\partial}{\partial\omega}\left(f(t)e^{-i\omega t}\right)dt = \int_{-\infty}^{\infty} itf(t)e^{-i\omega t}dt = -i\mathcal{F}(tf(t))$$

So the fourier transform of $tf(t)$ is $\mathcal{F}(tf(t)) = id\tilde{f}(\omega)/d\omega$. ∎

(c) The Fourier transform of $f(mt+c)$ is

$$\frac{e^{i\omega c/m}}{m}\tilde{f}\left(\frac{\omega}{n}\right)$$

Solution:
Making a change of variable $mt + c = p$; $\quad t = \frac{p-c}{m}$; $\quad dt = \frac{1}{m}dp$ so $e^{-i\omega t} = e^{i\omega c/m}e^{-i\omega/mp}$

$$\mathcal{F}(f(mt+c)) = \int_{-\infty}^{\infty} f(mt+c)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(p)e^{i\omega c/m}e^{i\omega p}\frac{1}{m}dp = \frac{e^{i\omega c/m}}{m}\int_{-\infty}^{\infty} f(p)e^{-i\omega/mp}dp = \frac{e^{i\omega c/m}}{m}\tilde{f}\left(\frac{\omega}{m}\right)$$

So the fourier transform of $f(mt+c)$ is shown as required. ∎

2. Find the fourier sine transform $\tilde{f}(\omega)$ of of the function $f(t) = t^{-1/2}$ and by differentiating with respect to ω find the differential equation satisfied by it. Hence show that the the sine transform of this function is the function itself.

Solution:
By definition of sine transform $\tilde{f}(\omega) = \int_0^\infty f(t)\sin(\omega t)dt$ we have for $f(t) = t^{-1/2}$.

$$\frac{d}{d\omega}(\tilde{f}(\omega)) = \frac{d}{d\omega}\int_0^\infty \frac{1}{\sqrt{t}}\sin(\omega t)dt = \int_0^\infty \frac{\partial}{\partial\omega}\left(\frac{1}{\sqrt{t}}\sin(\omega t)\right)dt = \int_0^\infty \sqrt{t}\cos(\omega t)dt$$

Integrating the RHS by parts we get

$$\frac{d}{d\omega}(\tilde{f}(\omega)) = \sqrt{t}\frac{sin(\omega t)}{\omega}\bigg|_0^\infty - \int_0^\infty \frac{1}{2\sqrt{t}}\frac{sin(\omega t)}{\omega} = \frac{1}{\omega}\left[\lim_{t\to\infty}\overbrace{\sqrt{t}\sin(\omega t)}^{0 \text{ assumed}} - 0\right] - \frac{1}{2\omega}\tilde{f}(\omega)$$

So the differential equation satisfied by the sine transform is

$$\frac{d}{d\omega}\left(\tilde{f}(\omega)\right) + \frac{1}{2\omega}\tilde{f}(\omega) = 0$$

This differntial equation can be solved as:

$$\frac{d\tilde{f}(\omega)}{d\omega} = -\frac{1}{2\omega}\tilde{f}(\omega); \qquad \Rightarrow \int\frac{d\tilde{f}(\omega)}{\tilde{f}(\omega)} = \int -\frac{d\omega}{2\omega}; \qquad \Rightarrow \ln(\tilde{f}(\omega)) = -\frac{1}{2}\ln(\omega) + \ln A; \qquad \Rightarrow \tilde{f}(\omega) = A\omega^{-1/2}$$

But since $f(t) = t^{-1/2}$ the value of $f(\omega) = w^{-1/2}$, so from above expression we get.

$$\tilde{f}(\omega) = Af(\omega)$$

Since we have the sine transform $\tilde{f}(\omega) = Af(\omega)$ the sine transform fo this given function is the function itself. ∎

3. Prove the equality

$$\int_0^\infty e^{-2at}\sin^2 at\,dt = \frac{1}{\pi}\int_0^\infty \frac{a^2}{4a^4 + w^4}d\omega$$

Solution:
It can be noticed that the LHS of the given equality is the square integral of function $f(t) = e^{-at}\sin(at)$ from 0 to ∞. Since the lower limit is 0 we can take the fourier transform of this function $u(t)f(t)$ where $u(t)$ is the step function

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} u(t)f(t)e^{-i\omega t}dt = \int_0^\infty e^{-at}\sin(at)e^{-i\omega t}dt = \frac{a}{a^2 + (a + i\omega)^2}$$

The absolute value of the fourier transfom of the function is

$$\left|\tilde{f}(\omega)\right| = \left|\frac{a}{a^2 + (a + i\omega)^2}\right| = \frac{a^2}{\sqrt{4a^4 + w^4}}$$

Now by use of Parseval's theorem we have

$$\int_{-\infty}^{\infty} |u(t)f(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega \qquad \text{(Parseval's theorem)}$$

Substituting $f(t)$ and $\tilde{f}(\omega)$ noting that the function $\tilde{f}(\omega)$ is even

$$\int_0^\infty e^{-2at}\sin^2(at)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{a^2}{\sqrt{4a^4+w^4}}\right)^2 d\omega = \frac{1}{\pi}\int_0^\infty \frac{a^4}{4a^4+w^4}d\omega$$

This completes the proof. ■

4. By writing $f(x)$ as an integral involving the δ-function, $\delta(\xi - x)$ and taking the laplace transform of both sides show that the transform of the solution of the equation

$$\frac{d^4y}{dx^4} - y = f(x)$$

for which y and its first three derivatives vanish at $x = 0$ can be written as

$$\tilde{y}(s) = \int_0^\infty f(\xi)\frac{e^{-s\xi}}{s^4-1}d\xi$$

Solution:
The function $f(x)$ can be written as the integral of delta functions as

$$f(x) = \int_0^\infty \delta(\xi - x)f(\xi)d\xi$$

So the Laplace transform of the function is

$$\tilde{f}(s) = \int_0^\infty \left\{\int_0^\infty \delta(\xi - x)f(\xi)d\xi\right\}e^{-sx}dx = \int_0^\infty \left\{\int_0^\infty \delta(\xi - x)e^{-sx}dx\right\}f(\xi)d\xi = \int_0^\infty e^{-s\xi}f(\xi)d\xi$$

Taking the laplace transform of the given differential equation we get

$$s^4\tilde{y}(s) - \tilde{y}(s) = \tilde{f}(s) = \int_0^\infty e^{-s\xi}f(\xi)d\xi; \qquad \Rightarrow \tilde{y}(s) = \int_0^\infty \frac{e^{-s\xi}}{s^4-1}f(\xi)d\xi$$

Now for the solution this function can be expressed as the product of two functions as

$$\tilde{y}(s) = \underbrace{\frac{1}{s^4-1}}_{\tilde{g}(s)}\underbrace{\int_0^\infty f(\xi)e^{-s\xi}d\xi}_{\tilde{f}(s)}$$

The inverse laplace transform of $\tilde{f}(s)$ is simply $f(x)$ and the fourier transform of $\tilde{g}(s)$ can be obtained as

$$g(s) = \mathcal{L}^{-1}\left(\frac{1}{s^4-1}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right]\right) = \frac{1}{2}[\sinh(x) - \sin(x)]$$

Now the laplace inverse of the product of the function is the convolution of inverses so

$$y(x) = f(x) * g(x) = \int_0^x f(\xi)g(x-\xi)d\xi = \frac{1}{2}\int_0^x f(\xi)[\sinh(x-\xi) - \sin(x-\xi)]d\xi$$

Which completes the proof. ■