PHYS :501 Mathematical Physics

Homework #5

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1. Use contour integration to find the inverse Fourier transform $f(t)$ of the function

$$F(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$$

(where $a > 0$), for all values of t. Recall that F was obtained as a Fourier transform of a step function with a discontinuity at $|t| = a$. What is the value of $f(a)$? (Determine $f(a)$ from the integral – don't appeal to the integral properties of Fourier Transforms!).

Solution:

Writing it as

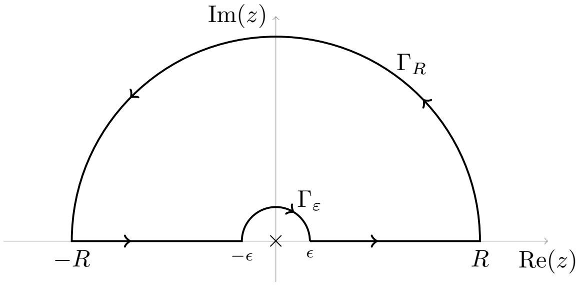
$$\begin{aligned} f(t) = \mathcal{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega}\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega}e^{-i\omega t}d\omega = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{e^{i\omega a}-e^{-i\omega a}}{2i\omega}e^{-i\omega t}d\omega \\ &= \frac{1}{2i\pi}\left[\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(a-t)\omega}}{\omega}d\omega}_{I_1} - \underbrace{\int_{-\infty}^{\infty}\frac{e^{-i(a+t)\omega}}{\omega}d\omega}_{I_2}\right] \\ &= \frac{1}{2i\pi}\left[I_1 - I_2\right] \end{aligned} \tag{1}$$

Considering the integral

$$A = \oint_C \frac{e^{i(a-t)z}}{z}dz = \int_{\Gamma_R}\frac{e^{i(a-t)z}}{z}dz + \int_{\Gamma_\epsilon}\frac{e^{i(a-t)z}}{z}dz + \underbrace{\int_{-R}^{-\epsilon}\frac{e^{i(a-t)\omega}}{\omega}d\omega + \int_{\epsilon}^{R}\frac{e^{i(a-t)}}{\omega}d\omega}_{I_1}$$

If we take limit as $R \to \infty$ and $\epsilon \to 0$ the last two terms of the integral converge to the integral along the ω axis. If the contour is in the upper half of the plane, then the first term of above integral goes to zero by Jordans Lemma if $(a - t) > 0$. But if $(a - t) < 0$ then the integral goes to zero only if the contour is in the lower half of the z plane

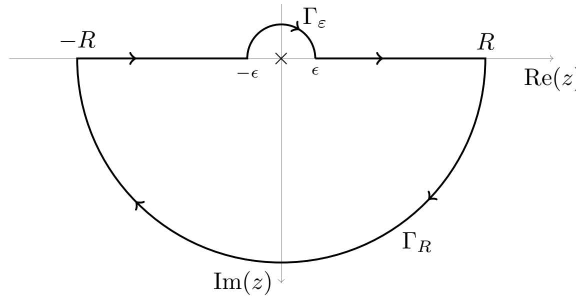
If $\mathbf{a - t > 0}$; $\mathbf{t < a}$



As seen above

$$
\begin{aligned}
A = I_1 + \underbrace{\int_{\Gamma_R} \frac{e^{i(a-t)z}}{z} dz}_{\text{0 By Jordan's Lemma}} + \int_{\Gamma_\epsilon} \frac{e^{i(a-t)z}}{z} dz = 0 \\
\Rightarrow I_1 - \frac{1}{2} 2\pi i \text{Res} f(0) = 0 \\
\Rightarrow I_1 - \pi i \lim_{z\to 0} z \frac{e^{i(a-t)z}}{z} = 0 \\
\Rightarrow I_1 = \pi i
\end{aligned} \tag{2}
$$

If $\mathbf{a - t < 0}$; $\mathbf{t > a}$



Also we can see

$$
\begin{aligned}
A = I_1 + \underbrace{\int_{\Gamma_R} \frac{e^{i(a-t)z}}{z} dz}_{\text{0 By Jordan's Lemma}} + \int_{\Gamma_\epsilon} \frac{e^{i(a-t)z}}{z} dz = -2\pi i \text{Res} f(0) \\
\Rightarrow I_1 - \frac{1}{2} 2\pi i \text{Res} f(0) = -2\pi i \text{Res} f(0) \\
\Rightarrow I_1 = -\pi i \text{Res} f(0) \\
\Rightarrow I_1 = -\pi i \lim_{z\to 0} z \frac{e^{i(a-t)z}}{z} = -\pi i
\end{aligned} \tag{3}
$$

If $t = a$ (with contour on top half,) then

$$
A = I_1 + \int_{\Gamma_R} \frac{e^{i(a-t)z}}{z} dz + \int_{\Gamma_\epsilon} \frac{e^{i(a-t)z}}{z} dz = I_1 + \underbrace{\int_{\Gamma_R} \frac{1}{z} dz}_{\pi i \text{Res} f(0)} + \underbrace{\int_{\Gamma_\epsilon} \frac{1}{z} dz}_{-\pi i \text{Res} f(0)} = 0; \Rightarrow I_1 = 0 \tag{4}
$$

From (**??**) and Eq. (**??**) and Eq. (**??**) we get

$$
I_1 = \begin{cases} \pi i & \text{if } a - t > 0 \\ 0 & \text{if } t = a \\ -\pi i & \text{if } a - t < 0 \end{cases} \tag{5}
$$

Considering the integral

$$
B = \oint_C \frac{e^{-i(a+t)z}}{z} dz = \int_{\Gamma_R} \frac{e^{-i(a+t)z}}{z} dz + \int_{\Gamma_\epsilon} \frac{e^{-i(a+t)z}}{z} dz + \underbrace{\int_{-R}^{-\epsilon} \frac{e^{-i(a+t)\omega}}{\omega} d\omega + \int_{\epsilon}^{R} \frac{e^{-i(a+t)}}{\omega} d\omega}_{I_2} = 0
$$

By similar arguments

$$
I_2 = \begin{cases} \pi i & \text{if } a + t < 0 \\ 0 & \text{if } a + t = 0 \\ -\pi i & \text{if } a + t > 0 \end{cases} \tag{6}
$$

From Eq. (**??**) and Eq. (**??**) we get.

$$\begin{aligned}
&\text{if } t<-a; && I_1=\pi i \text{ and } I_2=\pi i \Rightarrow I_1-I_2=0; \quad f(t)=\frac{1}{2\pi i}\left[I1-I2\right]=0\\
&\text{if } t=-a; && I_1=\pi i \text{ and } I_2=0 \Rightarrow I_1-I_2=\pi i; \quad f(t)=\frac{1}{2\pi i}\left[I1-I2\right]=\frac{1}{2}\\
&\text{if } -a<t<a; && I_1=\pi i \text{ and } I_2=-\pi i \Rightarrow I_1-I_2=2\pi i; \quad f(t)=\frac{1}{2\pi i}\left[I1-I2\right]=1\\
&\text{if } t=a; && I_1=0 \text{ and } I_2=-\pi i \Rightarrow I_1-I_2=\pi i; \quad f(t)=\frac{1}{2\pi i}\left[I1-I2\right]=\frac{1}{2}\\
&\text{if } t>a; && I_1=-\pi i \text{ and } I_2=-\pi i \Rightarrow I_1-I_2=0; \quad f(t)=\frac{1}{2\pi i}\left[I1-I2\right]=0
\end{aligned}$$

Combining all these we get

$$f(t)=\begin{cases}1 & |t|<a\\ \frac{1}{2} & |t|=a\\ 0 & |t|>a\end{cases}$$

So the value of $f(a)$ is $\frac{1}{2}$ from the inverse fourier transform. ■

2. Find the $3-D$ Fourier transform of the wave function for a $2p$ electron in a hydrogen atom:

$$\psi(\vec{x})=(32\pi a^5)^{-1/2}ze^{-r/2a_0}$$

where $a=\frac{\hbar^2}{me^2}$ is the Bohr radius, r is radius, and z is a rectangular corrdinate.
Solution:
Supposing $A=(32\pi a^5)^{-1/2}$ and in spherical coordinate system $z=r\cos(\theta)$. Also the volume element in spherical system is $d^3r=r^2\sin(\theta)d\phi d\theta$ Also *due to spherical symmetry we can write* $\vec{k}\cdot\vec{r}=kr\cos(\theta)$*[Riley and Hobson pp 906]* The fourier transform is then

$$\Psi(k)=\frac{1}{\sqrt{(2\pi)^3}}\int_0^\infty rcos(\theta)e^{r/2a}e^{-ikr\cos(\theta)}d^3r=\frac{2\pi}{\sqrt{(2\pi)^3}}\int_0^\pi\int_0^\infty r^3e^{r/2a}\sin(\theta)\cos(\theta)e^{ikr\cos(\theta)}d\theta dr$$

$$\Psi(k)=\frac{A}{\sqrt{(2\pi)^3}}\int_0^\infty dr\,r^3\,e^{-r/(2a)}\int_0^\pi d\theta\,\sin\theta\cos\theta\,e^{ikr\cos\theta}=\frac{A}{\sqrt{(2\pi)^3}}\int_0^\infty dr\,r^3\,e^{-r/(2a)}\int_\pi^0 d(\cos\theta)\cos\theta e^{ikr\cos\theta}$$

Supposing $krcos(\theta)=u\ du=sin(\theta)kdr$

$$\begin{aligned}
&=\frac{A}{\sqrt{(2\pi)^3}}\int_0^\infty dr\,r^3\,e^{-r/(2a)}\left[-i\frac{\partial}{\partial(kr)}\int_{-1}^1 du\,e^{ikru}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty dr\,r^3\,e^{-r/(2a)}\frac{\partial}{\partial(kr)}\frac{\sin kr}{kr}\\
&=\sqrt{\frac{2}{\pi}}\int_0^\infty dr\,r^3\,e^{-r/(2a)}\left[\frac{\cos kr}{kr}-\frac{\sin kr}{(kr)^2}\right]
\end{aligned}$$

The integral of this function can be obtained with contour integral
With substitution $z=1/(2a)-ik$ and $cos(kr)=\mathrm{Re}e^{ikr}$

$$\begin{aligned}
\Psi(k)&=A\sqrt{\frac{2}{\pi}}\frac{2}{k}\,\mathrm{Re}\left[\frac{1}{\left(\frac{1}{2a}-ik\right)^3}\right]-A\sqrt{\frac{2}{\pi}}\frac{1}{k^2}\,\mathrm{Im}\left[\frac{1}{\left(\frac{1}{2a}-ik\right)^2}\right]\\
&=A\sqrt{\frac{2}{\pi}}\frac{2}{k}\frac{\frac{1}{8a^3}-\frac{3k^2}{2a}}{\left(\frac{1}{4a^2}+k^2\right)^3}-A\sqrt{\frac{2}{\pi}}\frac{1}{k^2}\frac{\frac{k}{a}}{\left(\frac{1}{4a^2}+k^2\right)^2}\\
&=-A\sqrt{\frac{2}{\pi}}256a^4\frac{ka}{(1+4k^2a^2)^3}
\end{aligned}$$

This gives the fourier transform of the function. ■

3. Consider the solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + xy = 0$$

for which $|y| \to 0$ as $|x| \to \infty$. (This is the *Airy equation.* It appears in the theory of the diffaraction of light.)

(a) Sketch the solution. Don't use Mathematica!. Specifically, what behavior do you expect as $x \to -\infty$ and $x \to +\infty$?

Solution:

■

(b) By fourier transforming the above equation, determine $Y(\omega)$, te Fourier transform of $y(x)$, and hence write down an integral expression for $y(x)$. (Hint: What is the inverse transform of $Y'(\omega)$)

Solution:

Let us suppose that the fourier transform of $y(x)$ is $Y(\omega)$. The fourier transform can be written as

$$\mathcal{F}(y(x)) = Y(\omega) = \int_{-\infty}^{\infty} y(x)e^{-i\omega x}dx$$

.

Taking derivative of both sides with respect to ω

$$Y'(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} y(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} y(x)\frac{\partial}{\partial\omega}\left(e^{-i\omega x}\right)dx = -i\int_{-\infty}^{\infty} xy(x)e^{-i\omega x}dx = -i\mathcal{F}(xy) \tag{7}$$

Taking the fourier transform of both sides of given differential equation we get,

$$y''(x) + xy = 0; \Rightarrow \mathcal{F}\left[\frac{d^2y}{dx^2}\right] + \mathcal{F}(xy) = \mathcal{F}(0);$$

Using the property of fourier transform $\mathcal{F}(y'') = (-i\omega)^2\mathcal{F}(y)$ the fourier transform and using $\mathcal{F}(xy)$ from Eq. (**??**) we get.

$$(-i\omega)^2Y(\omega) - iY'(\omega) = 0; \Rightarrow \frac{Y'(\omega)}{Y(\omega)} = -i\omega^2; \Rightarrow \int \frac{Y'(\omega)}{Y(\omega)}d\omega = \int -i\omega^2 d\omega; \Rightarrow Y(\omega) = e^{-i\frac{\omega^3}{3}}$$

The solution for the Airy equation which is our original differential equation is just the inverse fourier transform of this equation.

$$y(x) = \mathcal{F}^{-1}\left[Y(\omega)\right] = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{\frac{-i\omega^3}{3}} e^{-i\omega x} d\omega$$

This gives the integral expression for the solution of the differential equation required. ■

4. Find the Greens function $G(x, x')$ for the equation

$$\frac{d^2y}{dx^2} - k^2 y = f(x)$$

for $0 \leq x \leq l$, with $y(0) = y(l) = 0$.
Solution:
The green's function solution to non homogenous differential equation $\mathcal{L}y = f(x)$ is a solution to homogenous part of the differential equation with the source part replaced as delta function $\mathcal{L}y = \delta(x - x')$. The ontained solution is $G(x, x')$, i.e., $\mathcal{L}G(x, x') = \delta(x - x')$. This solution corresponds to the homogenous part only as it is independent of any source term $f(x)$.

$$\frac{d^2}{dx^2}G(x, x') - k^2 y = \delta(x - x'); \qquad \text{with } G(0, x') = 0; \text{ and } G(l, x') = 0 \text{ for all } 0 \leq x' \leq l \tag{8}$$

Since delta function $\delta(x - x')$ is zero everywhere except $x = x'$ we can find solution for two regions $x < x'$ and $x > x'$. For $x < x'$ let the solution to $\mathcal{L}y = 0$ be $y_1(x)$ and for $x > x'$ be $y_2(x)$ then

$$y_1''(x) - k^2 y_1(x) = 0; \text{ for } x < x'; \qquad y_2''(x) - k^2 y_2(x) = 0; \text{ for } x > x'$$

These are well known harmonic oscillator equations whose solution are

$$y_1(x) = Asin(kx) + Bcos(kx); \qquad y_2(x) = Csin(kx) + Dcos(kx)$$

By the boundary condition $y_1(0) = 0$ and $y_2(l) = 0$. These immediately imply that $B = 0$. Also since the soution to the differential equation must be continuous $y_1(x') = y_2(x')$. Also integrating Eq. (**??**) in the vicinity of x' we get

$$y'(x)\Bigg|_{x'_-}^{x'_+} - k^2 \underbrace{\int y dx\Bigg|_{x'_-}^{x'_+}}_{0 \text{ By contunity}} = \int_{x'_-}^{x'_+} \delta(x - x')dx; \Rightarrow y'(x'_+) - y'(x'_-) = 1$$

From three different conditions, (i) contunity at x', (ii) $y_2(l) = 0$ and (iii) $y_1'(x') - y_2'(x') = 1$ we get following three linear equations. Using there parameters we get.

$$\begin{aligned} Ck\cos(kx') - Dk\sin(kx') - Ak\cos(kx') &= 1 \\ C\sin(kx') + D\cos(kx') - A\sin(kx') &= 0 \\ C\sin(kl) + D\cos(kl) &= 0 \end{aligned}$$

Which can be written in the matrix form and solved as.

$$\begin{bmatrix} k\cos(kx') & -k\sin(kx') & -k\cos(kx') \\ \sin(kx') & \cos(kx') & -\sin(kx') \\ \sin(kl) & \cos(kl) & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ A \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C \\ D \\ A \end{bmatrix} = \begin{bmatrix} \frac{\sin(kx')}{k\tan(kl)} \\ -\frac{1}{k}\sin(kx') \\ -\frac{\sin(k(l-x'))}{k\sin(kl)} \end{bmatrix}$$

Giving

$$C = \frac{\sin(kx')}{k\tan(kl)}; \qquad D = -\frac{1}{k}\sin(kx'); \qquad A = -\frac{\sin(k(l-x'))}{k\sin(kl)}$$

So the required function is

$$G(x,x') = \begin{cases} y_1(x) = -\frac{\sin\left(k\left(l-x'\right)\right)}{k\sin(kl)}\sin(kx) & \text{if } x < x' \\ y_2(x) = \frac{sin(kx')}{k}\left(\frac{\sin(kx)}{\tan(kl)} - \cos(kx)\right) & \text{if } x > x' \end{cases} \tag{9}$$

Eq.(**??**) gives the Green's function whcin can be used to find the solution to the differential equation

$$y(x) = \int G(x,x')f(x')dx'$$

The solution to the original inhomogenous differential equation can is given by the above expression in terms of Green's function. ■

5. Poissons equation (in three dimensions) is $\nabla^2\phi = 4\pi G\rho$

(a) Let $\widetilde{\phi}(\vec{k})$ be the fourier transforms of $\phi(x)$ and $\rho(x)$, respectively show that:

$$\widetilde{\phi} = -\frac{4\pi G\widetilde{\rho}}{k^2}$$

and hence write down an integral expression for $\phi(x)$.
Solution:
Taking the fourier transform ov Poissons equation we have

$$4\pi G\mathcal{F}(\rho(r)) = \int\limits_{-\infty}^{\infty} \nabla^2\phi(r)e^{ik\cdot r}d^3r$$

Wriging in cartesian coordinate system $\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ we have

$$\begin{aligned} 4\pi G\widetilde{\rho}(\vec{k}) &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \phi(\vec{r})\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)e^{xk_x+yk_y+zk_z}dxdydz \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \phi(\vec{r})\left((ik_x)^2 + (ik_y)^2 + (ik_z)^2\right)e^{i\vec{k}\cdot\vec{r}}dxdydz \\ &= (-k_x^2 - k_y^2 - k_z^2)\mathcal{F}(\phi(\vec{r})) = -\left|\vec{k}\right|^2\widetilde{\phi}(\vec{k}); \\ \Rightarrow \widetilde{\phi}(\vec{k}) &= -\frac{4\pi G\widetilde{\rho}(\vec{k})}{k^2} \end{aligned}$$

This gives the expression for the fourier transform for Poisson's equation. This can be used to get the expression of $\phi(\vec{x})$ which is

$$\phi(\vec{x}) = \frac{4\pi G}{\sqrt{(2\pi)^3}}\int\limits_{-\infty}^{\infty} \frac{1}{k^2}\tilde{\rho}(\vec{k})e^{-i\vec{k}\cdot\vec{r}}d^3k \tag{10}$$

This is the expression for $\phi(x)$ which is the solution to Poisson's equation. ■

(b) For a point mass at the origin, $\rho(x) = m\delta(x)$. Use the above to determine the expression for $\phi(x)$
Solution:
Taking the fourier transform of given density function

$$\tilde{\rho}(k) = \int_{-\infty}^{\infty} m\delta(\vec{r})e^{i\vec{k}\cdot\vec{r}}d^3r = m; \qquad \text{Integral of delta function is 1}$$

Substuting this in Eq. (**??**) we get

$$\phi(\vec{x}) = \frac{4\pi G}{\sqrt{(2\pi)^3}} \int_{-\infty}^{\infty} \frac{m}{k^2} e^{-i\vec{k}\cdot\vec{r}} d^3k = \frac{4\pi Gm}{\sqrt{(2\pi)^3}} \int_{-\infty}^{\infty} k^{-2} e^{-i\vec{k}\cdot\vec{r}} d^3k$$

This integral should give $\phi(x) = -\frac{Gm}{x}$ for $x > 0$ ■