PHYS :501 Mathematical Physics

Homework #4

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1. Use contour integration to compute the integral

$$I = \int_{-1}^{1} \frac{dx}{(a^2+x^2)\sqrt{1-x^2}}$$

where a is real and the integrand has a branch cut running from -1 to 1. Sketch the contour you have chosen and carefully justify your reasoning to evaluate or neglect each portion of the total integral.
Solution:
We can write the above integral as

$$\oint \frac{dz}{(a^2+z^2)\sqrt{1-z^2}}$$

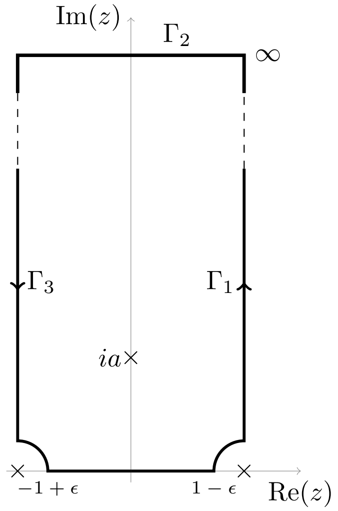


Figure 1: There are poles at ± 1 and $\pm ia$ .Since the function is even the integral along two vertical lines will be equal and opposite and vanish. The integral along the bottom horizontal line is what we want, and the integral along the top horizontal line will vanish because at large value of z; $\frac{1}{(a^2+z^2)\sqrt{1-z^2}} = 0$. In the closed contour integral only leaves the integral along the x axis from -1 to 1.

$$\oint f(z)dz = I - 2\pi i \frac{1}{4}\text{Res}(1) + \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \int_{\Gamma_3} f(z)dz - 2\pi i \frac{1}{4}\text{Res}(-1) \tag{1}$$

$$\text{Res}f(-1) = \lim_{z\to -1} -\frac{1-z}{(a^2+z^2)\sqrt{1-z^2}} = \lim_{z\to -1} -\frac{\sqrt{1-z}}{(a^2+z^2)\sqrt{1+z}} = 0$$

$$\text{Res}f(1) = \lim_{z\to 1} -\frac{1+z}{(a^2+z^2)\sqrt{1-z^2}} = \lim_{z\to 1} \frac{\sqrt{1+z}}{(a^2+z^2)\sqrt{1+z}} = 0$$

The only terms left in the RHS of Eq. (1) is I

$$\begin{aligned} I = \oint \frac{dz}{(a^2+z^2)\sqrt{1-z^2}} &= 2\pi i \text{Res} f(ia) \\ &= 2\pi i \lim_{z\to ia} \frac{\cancel{z-ia}}{(z+ia)(\cancel{z-ia})\sqrt{1-z^2}} \\ &= \frac{2\pi i}{2ia\sqrt{1+a^2}} \\ &= \frac{\pi}{a\sqrt{1+a^2}} \end{aligned}$$

So the required integral is $\int\limits_{-1}^{1} \frac{dx}{(a^2+x^2)\sqrt{1-x^2}} = \frac{\pi}{a\sqrt{1+a^2}}$. ■

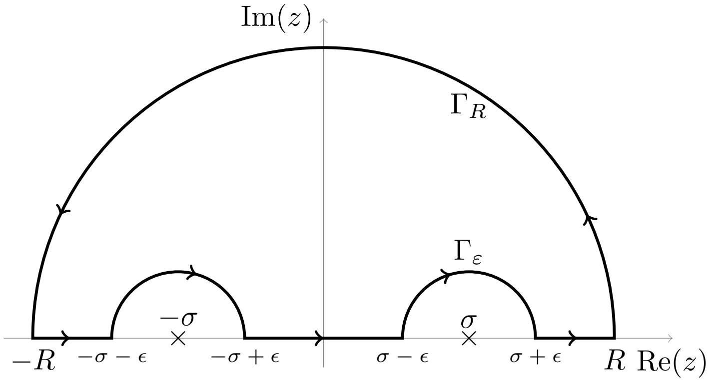
2. Work out the details of the contour integral in the context of quantum scattering problem. The problem involves evaluating the integral

$$I(\sigma) = \int\limits_{-\infty}^{\infty} \frac{x \sin x dx}{x^2 - \sigma^2}$$

The integrand has poles on the real axis, and so is only defined as a Cauchy Principal value, deforming the path of integration to avoid the poles using small semicircles of radius ϵ centered on $x = \pm\sigma$. State clearly the assumptions you make and the contours you choose, and show that

$$I(\sigma) = \pi \cos \sigma.$$

Solution:
There are two singular points at $\pm\sigma$. If we write the function as



$$f(z) = \frac{ze^{iz}}{z^2 - \sigma^2}; I(\sigma) = Im\left[\oint f(z)dz\right]$$

Taking this contour, the, integral along the big semicircular contour will go to zero by Jordan's Lemma. The integral along the line includes two semicircular hops.

$$\begin{aligned}
\int_{-R}^{R} f(z)dz &= \cancelto{-\xi}{\int_{-R}^{-\sigma-\epsilon}} f(z)dz + \cancelto{2\pi i\frac{1}{2}\text{Res}(-\sigma)}{\int_{-\sigma-\epsilon}^{-\sigma+\epsilon}} f(z)dz + \cancelto{0,}{\int_{-\sigma+\epsilon}^{\sigma-\epsilon}} f(z)dz + \cancelto{2\pi i\frac{1}{2}\text{Res}(\sigma)}{\int_{\sigma-\epsilon}^{\sigma+\epsilon}} f(z)dz + \cancelto{\xi}{\int_{\sigma+\epsilon}^{\sigma-\epsilon}} f(z)dz \\
&= \frac{2\pi i}{2}\left(\text{Res}(-\sigma) + \text{Res}(\sigma)\right) \\
&= \frac{2\pi i}{2}\left[\lim_{z\to-\sigma}\left(\frac{ze^{iz}}{z-\sigma}\right) + \lim_{z\to-\sigma}\left(\frac{ze^{iz}}{z+\sigma}\right)\right] \\
&= \pi i\left[\left(\frac{-\sigma e^{-i\sigma}}{-2\sigma}\right) + \left(\frac{\sigma e^{i\sigma}}{2\sigma}\right)\right] \\
&= \pi i\left[\left(\frac{e^{-i\sigma}}{2}\right) + \left(\frac{e^{i\sigma}}{2}\right)\right] \\
&= \pi i\cos(\sigma)
\end{aligned}$$

Our original integral was $I(\sigma) = Im\left[\oint f(z)dz\right]$ $=\text{Im}[\pi i\cos(\sigma)] = \pi\cos(\sigma)$. ■

3a) Find the series solution of the equation

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0$$

that is regular at $x = 0$. Under what circumstances (for what values of n) does the series converge for *all* x?

Solution:

Let the solution be $y(x) = \sum\limits_{r=0}^{\infty} a_r x^{r+k}$; where $a_0 \neq 0$. Then the first two derivatives are.

$$y'(x) = \sum_{r=0}^{\infty}(r+k)a_r x^{r+k-1}; \qquad y''(x) = \sum_{r=0}^{\infty}(r+k)(r+k-1)a_r x^{r+k-2}$$

Substuting these back into the given differential equation we get.

$$\sum_{r=0}^{\infty}(r+k)(r+k-1)a_r x^{r+k-2} - \sum_{r=0}^{\infty}(r+k)(r+k-1)a_r x^{r+k} - \sum_{r=0}^{\infty}(r+k)a_r x^{r+k} + \sum_{r=0}^{\infty}n^2 a_r x^{r+k} = 0$$

If we take out two terms from the summation sign in the first expression, we get

$$k(k-1)a_0x^{k-2} + k(k+1)a_1x^{k-1} + \sum_{r=2}^{\infty}(r+k)(r+k-1)a_r x^{r+k-2} - \sum_{r=0}^{\infty}(r+k)(r+k-1)a_r x^{r+k} - \sum_{r=0}^{\infty}(r+k)a_r x^{r+k} + \sum_{r=0}^{\infty}n^2 a_r x^{r+k} = 0$$

Since r is a dummy index $\sum\limits_{r=2}^{\infty}(r+k)(r+k-1)a_r x^{r+k-2}$ can be written as $\sum\limits_{r=0}^{\infty}(r+k+2)(r+k+1)a_{r+2}x^{r+k}$

$$k(k-1)a_0x^{k-2} + k(k+1)a_1x^{k-1} + \sum_{r=0}^{\infty}\left[(r+k+2)(r+k+1)a_{r+2} - (r+k)(r+k-1)a_r - (r+k)a_r x^{r+k} + n^2a_r\right]x^{r+k} = 0$$

Since we are expecting solution that is to be true for everyvalue of x every coefficient of each x^{r+k} should go to zero. If it didn't then we would have a polynomial of degree $r+k$ which would give $r+k$ solutions for x and would not be true for any general x other then the solution to it.

Equating the coefficient of x^{k-2} to zero we get $k(k-1)a_0 = 0; \Rightarrow k = \{0, 1\}$.
If we choose $k = 0$ then the coefficient of x^{k-1} which is $k(k+1)a_1$ goes to zero. So a_1 can be any arbitrary number. If we choose $k = 1$ then the coefficient of x^{k-1} which is $k(k+1)a_1 = 0$ requires that $a_1 = 0$. So

$$a_1 = \begin{cases} \text{arbitrary} & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k = 1 \end{cases}$$

Also the coefficient of x^{r+k} should be zero for every value of $r \geq 0$. Equating the coefficient of $x^{r+k} = 0$ we get

$$(r+k+2)(r+k+1)a_{r+2} = \left((r+k)^2 - n^2\right) a_r; \qquad \Rightarrow a_{r+2} = \frac{(r+k)^2 - n^2}{(r+k+2)(r+k+1)} a_r$$

for k = 0

$$\begin{aligned} a_{r+2} &= \frac{r^2 - n^2}{(r+2)(r+1)} a_r \\ a_2 &= \frac{-n^2}{2!} a_0 \\ a_3 &= \frac{1 - n^2}{3!} a_1 \\ a_4 &= \frac{2^2 - n^2}{4 \cdot 3} a_2 = \frac{n^2(n^2 - 2^2)}{4!} a_0 \\ a_5 &= \frac{3^2 - n^2}{5 \cdot 4} a_1 = \frac{(n^2 - 1)(n^2 - 3^2)}{5!} a_1 \\ &\vdots \qquad \vdots \end{aligned}$$

The solution then is

$$\begin{aligned} y_0(x) = a_0 &\left\{ 1 - \frac{n^2}{2!} x^2 + \frac{n^2(n^2 - 2^2)}{4!} x^4 + \cdots \right\} \\ + a_1 &\left\{ x - \frac{n^2 - 1}{3!} x^3 + \frac{(n^2 - 3^2)(n^2 - 1)}{5!} x^5 + \cdots \right\} \end{aligned}$$

for k=1

$$\begin{aligned} a_{r+2} &= \frac{(r+1)^2 - n^2}{(r+3)(r+2)} a_r; \qquad a_1 = 0 \\ a_2 &= \frac{1 - n^2}{3!} a_0 \\ a_3 &= \frac{2^2 - n^2}{3!} a_1 = 0 \\ a_4 &= \frac{3^2 - n^2}{5 \cdot 4} a_2 = \frac{(n^2 - 1)(n^2 - 3^2)}{5!} a_0 \\ a_5 &= 0 \\ &\vdots \qquad \vdots \end{aligned}$$

The solution then is

$$y_1(x) = a_0 \left\{ x - \frac{n^2 - 1}{3!} x^3 + \frac{(n^2 - 3^2)(n^2 - 1)}{5!} x^5 + \cdots \right\}$$

The two solution obtained above are linearly dependent so, we will analyze convergence for the first solution. $y_0(x)$ has a form of

$$y_0(x) = a_0\{\text{Even Function of } x\} + a_1\{\text{Odd Function of } x\}$$

Foe n = Even Integer, the Even function will be a n^{th} degree polynoial and similarly for n being odd.
For the convergence of series, we get from the recurrence relation,

$$\lim_{r\to\infty} \frac{t_{r+2}}{t_r} = \lim_{r\to\infty} \frac{a_{r+2}}{a_r} x^2 = \lim_{r\to\infty} \frac{r^2 - n^2}{(r+2)(r+1)} x^2 = x^2$$

For convergence $\lim\limits_{r\to\infty} \frac{t_{r+2}}{t_r} < 1$ which implies that $x^2 < 1; \Rightarrow |x| < 1$ For this series to converge for all values of x, the above ratio should be less than 1 for some value of n, but it doesn't happen for any n. So the series can't be convergent for all values of x. ■

(b) Find the series solution of the equation

$$4x^2y'' + (1 - p^2)y = 0$$

Solution:
Let the solution be $\sum\limits_{r=0}^{\infty} a_r x^{r+k}$ where $a \neq 0$. The Second derivative is

$$y'(x) = \sum_{r=0}^{\infty} (r+k) a_r x^{r+k-1}; \qquad y''(x) = \sum_{r=0}^{\infty} (r+k)(r+k-1) a_r x^{r+k-2}$$

Substuting these back into the given differential equation we get.

$$\sum_{r=0}^{\infty} 4(r+k)(r+k-1) a_r x^{r+k} + \sum_{r=0}^{\infty} (1-p^2) a_r x^{r+k} = 0; \quad \Rightarrow \sum_{r=0}^{\infty} \left[4(r+k)(r+k-1)a_r + (1-p^2)a_r\right] x^{r+k} = 0$$

Since we seek the solution of differential equation which is true for every value of x, it requires that every coefficient of x^{r+k} vanish.

$$\left\{4(r+k)(r+k-1) + (1-p^2)\right\} a_r = 0; \qquad \text{for } r \geq 0$$

Since we suppose $a_0 \neq 0$,

$$4(k+r)(k+r-1) + (1-p^2) = 0; \Rightarrow 4(k+r)^2 - 4(k+r) + (1-p^2) = 0;$$

The solution to the quadratic equation in k has the solution

$$k + r = \frac{4 \pm \sqrt{4^2 - 4 \cdot 4(1-p^2)}}{2 \cdot 4}; \qquad \Rightarrow k + r = \frac{1}{2}(1 \pm p)$$

Putting back the value of $x + r$ in our original solution we get,

$$y(x) = \sum_{r=0}^{\infty} a_r x^{r+k} = \sum_{r=0}^{\infty} a_r x^{\frac{1}{2}[1 \pm p]} = \left(\sum_{r=0}^{\infty} a_r\right) x^{\frac{1}{2}[1 \pm p]} = \xi x^{\frac{1}{2}[1 \pm p]}; \quad \text{Where } \xi = \sum_{r=0}^{\infty} a_r (\text{Constant})$$

So the two independent solution for the equation are $y(x) = \xi_1 x^{\frac{1}{2}(1+p)}$ and $y(x) = \xi_2 x^{\frac{1}{2}(1-p)}$. ■

(c) Given the one solution of the differential equation

$$y'' - 2xy' = 0$$

is $y(x) = 1$, use the Wronskian development to find a second, linearly independent solution. Describe the behavior near $x = 0$

Solution:
Comparing with $y'' + p(x)y' + q(x)y = 0$, $p(x) = -2x$ So,

$$\int p(x)dx = -x^2$$

We have $y_1(x) = 1$. The second solution is

$$\begin{aligned} y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1(x)^2} &= \int e^{x^2} dx \\ &= \int \left[1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots\right] dx \\ &= x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \cdots \end{aligned}$$

The function is well defined near $x = 0$. ■

4. A function $f(x)$ is periodic with period 2π, and can be written as a polynomial $P(x)$ for $-\pi < x < a$ and as a polynomial $Q(x)$ for $a < x < \pi$. Show that the Fourier coefficients c_n of f go to zero at least as fast as $^1/_{n^2}$ as $n \to \infty$if $P(a) = Q(a)$ and $P(\pi) = Q(\pi)$ (i.e. f is continuous), but only as $1/n$ otherwise.
Solution:
The fourier coefficeent is given by.

$$c_n = \int f(x)e^{-inx}dx = \int\limits_{-\pi}^{a} P(x)e^{-inx}dx + \int\limits_{a}^{\pi} Q(x)e^{-inx}dx$$

Integrating by parts

$$\begin{aligned} c_n &= \left[P(x)\frac{e^{-inx}}{-in}\right]_{-\pi}^{a} + \left[Q(x)\frac{e^{-inx}}{-in}\right]_{a}^{\pi} + \int P'(x)\frac{e^{-inx}}{in} + \int Q'(x)\frac{e^{-inx}}{in} \\ &= \frac{1}{n}\Bigg[(P(a) - Q(a))e^{-ina} + (Q(\pi)e^{-in\pi} - P(-\pi)e^{in\pi})\Bigg]^{(Q(\pi) - P(-\pi))\cos(n\pi))} + \int P'(x)\frac{e^{-inx}}{in} + \int Q'(x)\frac{e^{-inx}}{in} \\ &= \frac{1}{n}\left[(P(a) - Q(a))e^{-ina} + (Q(\pi) - P(-\pi))\cos(n\pi))\right] + \int P'(x)\frac{e^{-inx}}{in} + \int Q'(x)\frac{e^{-inx}}{in} \end{aligned}$$

If we continue on this way.

$$\begin{aligned} c_n =& \frac{1}{n}\left[(P(a) - Q(a))e^{-ia} + (Q(\pi) - P(-\pi))\cos(n\pi))\right] + \\ & \frac{1}{n^2}\left[(P'(a) - Q'(a))e^{-ia} + (Q'(\pi) - P'(-\pi))\cos(n\pi))\right] + \dots + \frac{1}{n^{r+1}}\int P^{(r)}(x)\frac{e^{-inx}}{in} + \int Q^{(r)}(x)\frac{e^{-inx}}{in} \end{aligned} \tag{2}$$

Let the order of polynomials $P(x)$ and $Q(x)$ be k_1 and k_2 respectively, are polynomials the derivatives will terminate when $r > \max\{k_1, k_2\}$ We will then have a expression for c_n which is a polynomial of $\frac{1}{n}$
If $P(a) = Q(a)$ and $P(-\pi) = Q(\pi)$ the first term of the Eq. (2) will vanish and c_n goes at least as $\frac{1}{n^2}$. It can go faster to zero if also the derivatives are equal then second term goes away. If the function do not agree at the boundaries then the first term of the c_n does not vanish and c_n goes only as fast as $\frac{1}{n}$. ■

5a) Find the Fourier series $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$ for $-1 < x < 1$ for the sawtooth function

$$f(x) = \begin{cases} -1 - x & (-1 < x < 0) \\ 1 - x & (\ \ 0 < x < 1) \end{cases} \tag{3}$$

Solution:
The period of the function is $T = 2$, The fourier coefficient can be calculated as

$$\begin{aligned} b_n = \frac{2}{T}\int f(x)sin(n\pi x) &= -\int\limits_{-1}^{0}(1 + x)\sin(n\pi x)dx + \int\limits_{0}^{1}(1 - x)\sin(n\pi x)dx \\ &= -\left[-\frac{1}{n\pi} + \frac{\cos(n\pi)}{n\pi}\right] + \left[\frac{1}{n\pi} + \frac{\cos(n\pi)}{n\pi}\right] \\ &= \frac{2}{n\pi} \end{aligned}$$

So the series solution is $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x)$. ■

(b) Plot the partial sums $S_N(x) = \sum_{n=1}^{N} b_n \sin(n\pi x)$ of the series for $0 \leq x \leq 1$, in steps of $\delta x = 0.0005$, and $N = 1, 5, 10, 20, 50, 100$ and 500. What is the maximum overshoot of Fourier series in the case $N = 500$, and at what value of x does it occur?

Solution:

The maximum overshoot for $N = 500$ occurs at $x = 0.0020$ and the value of overshoot is 0.1790. ■

Figure 2: Partial Fourier series plot for Eq.(3) ($\sum_{n=1}^{N} b_n \sin(n\pi x)$) for different N with Max overshoot of O_m at x_s