PHYS 501: Mathematical Physics I

Homework #1

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1. Let a and b be any two vectors in a real linear vector space, and define $c = a + \lambda b$, where is real. By requiring that $c \geq 0$ for all λ, derive the Cauchy-Schwartz inequality

$$(a \cdot a)(b \cdot b) \geq (a, b)^2$$

When does equality hold? (Use only the general properties of the inner product. Do not assume that it is possible to write $a.b = |a||b| \cos(\theta)$

Solution:

Lets assume $c = a + \lambda b$, then by the definition of inner product we get $(c.c) \geq 0$

$$\begin{aligned}
c.c &= (a + \lambda b)(a + \lambda b) \geq 0 \\
&\Rightarrow ((a + \lambda b).a) + \lambda(a + \lambda b).b) \geq 0 && \text{Linearity of inner product} \\
&\Rightarrow (a.a) + \lambda(b.a) + \lambda(a.b) + \lambda^2(b.b) \geq 0 && \text{Linearity of inner product with } \lambda\text{real} \\
&\Rightarrow (a.a) + 2\lambda(a.b) + \lambda^2(b.b) \geq 0 && (a.b) = (b.a) \text{ for real vector space}
\end{aligned}$$

The above inequality is quadratic in λ which is real value. All the inner product map to real values in this real linear space. The equality of above expression is a quadratic equation of in r with real coefficient. Since the quadriatic equation lies wholely above real axis it can't have real solution. The condition for which is

$$\begin{aligned}
4(a.a)(b.b) &\leq (2(a.b))^2 \\
(a.a)(b.b) &\leq (a.b)^2.
\end{aligned}$$

Clearly the equality hold when the two vectors are identical. □

2. Let A be any square matrix, and define $B = e^A \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!}$ Prove that an eivenvector of A with eigenvalue λ is an eigenvector of B, with eigenvalue e^λ.

Solution:

Given λ is eigenvalue of A. Let the corresponding eivenvector be C. By defition

$$AC = \lambda C$$

Pre multiplying above relation with A we get.

$$A(AC) = A(\lambda C) \Rightarrow (AA)C = \lambda(AC) \Rightarrow A^2 C = \lambda\lambda C = \lambda^2 C$$

By induction we get

$$A^n C = \lambda^n C \tag{1}$$

Now lets operate the vector C by B

$$\begin{aligned}
BC &= \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right) C && \text{(Definition of B Given)}\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} A^n C && \text{(Distributive property of Matrix over Matrix)}\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n C && \text{(From (1))}\\
&= \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) C && \text{(Distributive property of scalar over matrix)}\\
&= e^{\lambda} C && \text{(Definition of } e\text{)}
\end{aligned}$$

Since $BC = e^{\lambda}C$, C is the eigenvector of B with eigenvalue e^{λ}. □ □

3. Consider the 4-dimensional vector space of polynomials of degree less than or equal to 3, on the range $1 \le x \le 1$, spanned by the basis set $\{1, x, x^2, x^3\}$.The inner product of two polynomials in this space is defined as

$$(f, g) = \int_{-1}^{1} |x| f(x) g(x) dx$$

Use GramSchmidt orthogonalization to construct two orthonormal basis sets, as follows:

(a) Start with the set as listed above and begin the procedure with the function 1, as in class.

(b) Rewrite the set as $\{x^2, x, 1, x^3\}$and begin the orthogonalization procedure starting with x^2

Write down the matrix representing the transformation from basis (i) to basis (ii), and demon- strate that it is orthogonal.

Solution:

Let us write the cross product table for the given basis functions.

$$(1 \cdot 1) = \int_{-1}^{1} |x| 1 \cdot 1 dx = \int_{-1}^{0} -x dx + \int_{0}^{1} x dx = \left.\frac{-x^2}{2}\right]_{-1}^{0} + \left.\frac{-x^2}{2}\right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

$$(1 \cdot x) = \int_{-1}^{1} |x| 1 \cdot x dx = \int_{-1}^{0} -x^2 dx + \int_{0}^{1} x^2 dx = \left.\frac{-x^3}{3}\right]_{-1}^{0} + \left.\frac{x^3}{3}\right]_{0}^{1} = -\frac{1}{3} + \frac{1}{3} = 0$$

$$(1 \cdot x^2) = \int_{-1}^{1} |x| 1 \cdot x^2 dx = \int_{-1}^{0} -x^3 dx + \int_{0}^{1} x^3 dx = \left.\frac{-x^4}{4}\right]_{-1}^{0} + \left.\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$(1 \cdot x^3) = \int_{-1}^{1} |x| 1 \cdot x^3 dx = \int_{-1}^{0} -x^4 dx + \int_{0}^{1} x^4 dx = \left.\frac{-x^5}{5}\right]_{-1}^{0} + \left.\frac{x^5}{5}\right]_{0}^{1} = -\frac{1}{5} + \frac{1}{5} = 0$$

$$(x \cdot 1) = (1 \cdot x) = 0$$

$$(x \cdot x) = \int_{-1}^{1} |x| x \cdot x dx = \int_{-1}^{1} |x| 1 \cdot x^2 dx = \frac{1}{2}$$

$$(x \cdot x^2) = \int_{-1}^{1} |x| x \cdot x^2 dx = \int_{-1}^{1} |x| 1 \cdot x^3 dx = 0$$

$$(x \cdot x^3) = \int_{-1}^{1} |x| x \cdot x^3 dx = \int_{-1}^{0} -x^5 dx + \int_{0}^{1} x^5 dx = \left.\frac{-x^6}{6}\right]_{-1}^{0} + \left.\frac{x^6}{6}\right]_{0}^{1} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$(x^2 \cdot 1) = (1 \cdot x^2) = \frac{1}{2} \qquad (x^2 \cdot x) = (x \cdot x^2) = 0$$

$$(x^2 \cdot x^2) = \int_{-1}^{1} |x| x^2 \cdot x^2 dx = \int_{-1}^{1} |x| x \cdot x^3 dx = \frac{1}{3}$$

$$(x^2 \cdot x^3) = \int_{-1}^{1} |x| x^2 \cdot x^3 dx = \int_{-1}^{0} -x^6 dx + \int_{0}^{1} x^6 dx = \left.\frac{-x^7}{7}\right]_{-1}^{0} + \left.\frac{x^7}{7}\right]_{0}^{1} = -\frac{1}{7} + \frac{1}{7} = 0$$

$$(x^3 \cdot 1) = (1 \cdot x^3) = 0 \qquad (x^3 \cdot x) = (x \cdot x^3) = \frac{1}{2} \qquad (x^3 \cdot x^2) = (x^3 \cdot x^2) = 0$$

$$(x^3 \cdot x^3) = \int_{-1}^{1} |x| x^3 \cdot x^3 dx = \int_{-1}^{0} -x^7 dx + \int_{0}^{1} x^7 dx = \left.\frac{-x^8}{8}\right]_{-1}^{0} + \left.\frac{x^8}{8}\right]_{0}^{1} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

For the first basis set: $1, x, x^2, x^3$; Let $(v_1, v_2, v_3, v_4) = (1, x, x^2, x^3)$ and let $(\hat{e_1}, \hat{e_4}, \hat{e_3}, \hat{e_4})$ be the corresponding orthonormal set.

$$e_1 = v_1 = 1 \qquad \hat{e_1} = \frac{e_1}{\sqrt{(e_1, e_1)}} = \frac{1}{\sqrt{(1 \cdot 1)}} = 1$$

$$e_2 = v_2 - (\hat{e_1}, v_2)\hat{e_1} \qquad \hat{e_2} = \frac{x}{\sqrt{(x, x)}} = \frac{1}{\sqrt{1/2}} = \sqrt{2}x$$

$$= x - (1, x)1 = x - 0 = x$$

The normal vector corresponding to v_3 can be found as.

$$\begin{aligned} e_3 &= v_3 - (\hat{e_2}, v_3)\hat{e_2} - (\hat{e_1}, v_3)\hat{e_1} \\ &= x^2 - (\sqrt{2}x, x^2)\sqrt{2}x - (1, x^2)1 \\ &= x^2 - \sqrt{2}(x, x^2)\sqrt{2}x - \frac{1}{2} \\ &= x^2 - \sqrt{2} \cdot 0 \cdot \sqrt{2}x - \frac{1}{2} \\ &= x^2 - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \hat{e_3} &= \frac{x^2 - 1/2}{\sqrt{(x^2 - 1/2, x^2 - 1/2)}} \\ &= \frac{x^2 - 1/2}{\sqrt{(x^2, x^2) - 2(x^2, 1/2) + (1/2, 1/2)}} \\ &= \frac{x^2 - 1/2}{\sqrt{\frac{1}{3} - 2 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4}}} \\ &= \sqrt{3}\left(2x^2 - 1\right) \end{aligned}$$

For $v_4 = x^2$ we similarly get:

$$\begin{aligned} e_4 &= v_4 - (\hat{e_3}, v_4)\hat{e_3} - (\hat{e_2}, v_4)\hat{e_2} - (\hat{e_1}, v_4)\hat{e_1} \\ &= x^3 - (\sqrt{3}\left(2x^2 - 1\right), x^3)\sqrt{3}\left(2x^2 - 1\right) - (\sqrt{2}x, x^3)\sqrt{2}x - (1, x^3)1 \\ &= x^3 - \sqrt{3}\left(2(x^2, x^3) - (1, x^3)\right)\sqrt{3}\left(2x^2 - 1\right) - \sqrt{2}(x, x^3)\sqrt{2}x \\ &= x^3 - \sqrt{3}\left(0 - 0\right)\sqrt{3}\left(2x^2 - 1\right) - \sqrt{2}\frac{1}{3}\sqrt{2}x \\ &= x^3 - 0 - \frac{2}{3}x \\ &= x^3 - \frac{2}{3}x \end{aligned}$$

$$\begin{aligned} \hat{e_3} &= \frac{x^3 - (2/3)x}{\sqrt{\left(x^3 - \frac{2}{3}x, x^3 - \frac{2}{3}x\right)}} \\ &= \frac{x^3 - (2/3)x}{\sqrt{\left((x^3, x^3) - 2\frac{2}{3}(x^3, x) - \frac{2}{3}^2(x, x)\right)}} \\ &= \frac{x^3 - (2/3)x}{\sqrt{\frac{1}{4} - 2 \cdot \frac{2}{3} \cdot 0 + \frac{2}{9}}} \\ &= 6\left(x^3 - \frac{2}{3}x\right) = 6x^3 - 4x \end{aligned}$$

Therefore the orthonormalized basis set is

$$\left\{1, \quad \sqrt{2}x, \quad \sqrt{3}(2x^2 - 1), \quad 6x^3 - 4x\right\}$$

Working out similarly,

Let $(\hat{e_1}', \hat{e_4}', \hat{e_3}', \hat{e_4}')$ be the corresponding orthonormal set.

$$e_1' = v_1 = x^2 \qquad \hat{e_1}' = \frac{e_1'}{\sqrt{(e_1', e_1')}} = \frac{x^2}{\sqrt{(x^2, x^2)}} = \frac{x^2}{\sqrt{1/3}} = \sqrt{3}x^2$$

$$\begin{aligned} e_2' &= v_2 - (\hat{e_1}', v_2) \\ &= x - (\sqrt{3}x^2, x)\sqrt{3}x^2 \\ &= x - \sqrt{3}\cdot 0 \cdot \sqrt{3}x^2 = x \end{aligned} \qquad \hat{e_2}' = \frac{e_2'}{\sqrt{(e_2', e_2')}} = \frac{x}{\sqrt{(x,x)}} = \frac{1}{\sqrt{1/2}} = \sqrt{2}x$$

$$\begin{aligned} e_3' &= v_3 - (\hat{e_2}', v_3)\hat{e_2}' - (\hat{e_1}', v_3)\hat{e_1}' \\ &= 1 - (\sqrt{2}x, 1)\sqrt{2}x - (\sqrt{3}x^2, 1)\sqrt{3}x^2 \\ &= 1 - \sqrt{2}(x, 1)\sqrt{2}x - \sqrt{3}(x^2, 1)\sqrt{3}x^2 \\ &= 1 - \sqrt{2}\cdot 0 \cdot \sqrt{2}x - \sqrt{3}\cdot \frac{1}{2} \cdot \sqrt{3}x^2 \\ &= 1 - \frac{3}{2}x^2 \end{aligned}$$

$$\begin{aligned} (e_3', e_3') &= \left(1 - 3/2\,x^2, 1 - 3/2\,x^2\right) \\ &= (1,1) - 2(1, -3/2\,x^2) + (-3/2\,x^2, -3/2\,x^2) \\ &= 1 + 2\,3/2\,(1, x^2) + (3/2)^2(x^2, x^2) \\ &= 1 + 2\cdot 3/2 \cdot 1/2 + (3/2)^2 0 = 4 \end{aligned}$$

$$\hat{e_3}' = \frac{e_3'}{\sqrt{(e_3', e_3')}} = \frac{1 - 3/2\,x^2}{\sqrt{4}} = 2 - 3x^2$$

$$\begin{aligned} e_4' &= v_4 - (\hat{e_3}', v_4)\hat{e_3}' - (\hat{e_2}', v_4)\hat{e_2}' - (\hat{e_1}', v_4)\hat{e_1}' \\ &= x^3 - (2 - 3x^2, x^3)(2 - 3x^2) - (\sqrt{2}x, x^3)\sqrt{2}x - (\sqrt{3}x^2, x^3)\sqrt{3}x^2 \\ &= x^3 - (2(1, x^3) - 3(x^2, x^3))(2 - 3x^2)) - \sqrt{2}(x, x^3)\sqrt{2}x - \sqrt{3}(x^2, x^3)\sqrt{3}x^2 \\ &= x^3 - (2\cdot 0 - 3\cdot 0)(2 - 3x^2)) - \sqrt{2}(1/3)\sqrt{2}x - \sqrt{3}(0)\sqrt{3}x^2 \\ &= x^3 - \sqrt{2}(1/3)\sqrt{2}x = x^3 - \frac{2}{3}x \end{aligned}$$

$$\begin{aligned} (e_4', e_4') &= \left(x^3 - 2/3\,x, x^3 - 2/3\,x\right) \\ &= (x^3, x^3) - 2\,2/3\,(x^3, x) + (2/3)^2(x, x) \\ &= 1/4 - 2\cdot 2/3 \cdot 0 + 2/9 \\ &= 1/36 \end{aligned}$$

$$\hat{e_4}' = \frac{e_4}{\sqrt{(e_4, e_4)}} = \frac{x^3 - 2/3\,x}{\sqrt{1/36}} = 6x^3 - 4x$$

So the orthonormal basis set is found to be.

$$\left\{\sqrt{3}x^2, \quad \sqrt{2}x, \quad -3x^2 + 2, \quad 6x^3 - 4x\right\}$$

We can now find the transformation matrix between these two basis set. If we suppose λ_{ij} be the elements of the transformation. Then $\hat{e}\ \lambda_{ij} = \left(\hat{e_i}, \hat{e_j}'\right)$ So,

$$\begin{aligned} \lambda_{11} &= \left(\hat{e_1}, \hat{e_1}'\right) = \left(1, \sqrt{3}x^2\right) = \sqrt{3}(1, x^2) = \sqrt{3}\,1/2 \\ \lambda_{12} &= \left(\hat{e_1}, \hat{e_2}'\right) = \left(1, \sqrt{2}x\right) = \sqrt{2}(1, x) = 0 \\ \lambda_{13} &= \left(\hat{e_1}, \hat{e_3}'\right) = \left(1, -3x^2 + 2\right) = -3(1, x^2) + 2(1, 1) = -3\,1/2 + 2 = 1/2 \\ \lambda_{14} &= \left(\hat{e_1}, \hat{e_4}'\right) = \left(1, 6x^3 - 4x\right) = 6(1, x^3) + 4(1, x) = 6\cdot 0 + 4\cdot 0 = 0 \\ \lambda_{21} &= \left(\hat{e_2}, \hat{e_1}'\right) = \left(\sqrt{2}x, \sqrt{3}x^2\right) = \sqrt{2\times 3}(x, x^2) = 0 \\ \lambda_{22} &= \left(\hat{e_2}, \hat{e_2}'\right) = \left(\sqrt{2}x, \sqrt{2}x\right) = \sqrt{2\times 2}(x, x) = 2\,1/2 = 1 \\ \lambda_{23} &= \left(\hat{e_2}, \hat{e_3}'\right) = \left(\sqrt{2}x, -3x^2 + 2\right) = -3\sqrt{2}(x, x^2) + \sqrt{2}(x, 1) = -3\sqrt{2}\cdot 0 + \sqrt{2}\cdot 0 = 0 \\ \lambda_{24} &= \left(\hat{e_2}, \hat{e_4}'\right) = \left(\sqrt{2}x, 6x^3 - 4x\right) = 6\sqrt{2}(1, x^3) + 4\sqrt{2}(1, x) = 6\sqrt{2}\cdot 0 + 2\sqrt{2}\cdot 0 = 0 \end{aligned}$$

Wroking this out we get the transformation matrix as.

$$\begin{pmatrix} \sqrt{3}{}^1\!/_2 & 0 & {}^1\!/_2 & 0 \\ 0 & 1 & 0 & 0 \\ {}^1\!/_2 & 0 & -\sqrt{3}{}^1\!/_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For an orthogonal matrix A, the inverse $A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. To prove that the matrix is orthogonal it suffices to show that the product of the matrix and its transpose is identity matrix.

$$\begin{pmatrix} \sqrt{3}{}^1\!/_2 & 0 & {}^1\!/_2 & 0 \\ 0 & 1 & 0 & 0 \\ {}^1\!/_2 & 0 & -\sqrt{3}{}^1\!/_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \sqrt{3}{}^1\!/_2 & 0 & {}^1\!/_2 & 0 \\ 0 & 1 & 0 & 0 \\ {}^1\!/_2 & 0 & -\sqrt{3}{}^1\!/_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $AA^T = I$, A^T is the inverse of matrix A which shows that the matrix A is orthogonal. □

4. (a) Transform the matrix A into a coordinate system in which A is diagonal, with the diagonal elements increasing from top to bottom. Write down the transformation matrix and the diagonalized A

$$A = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & i & 1 \end{pmatrix}$$

(b) A matrix B has real eigenvalues. Does it necessarily follow that B is hermitian? Prove the statement or give a counterexample.

Solution:

Lets find the eigenvalues of the matrix A. The determinant of $(A - \lambda I)$ is

$$\begin{vmatrix} -\lambda & -i & 0 & 0 & 0 \\ i & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & -i \\ 0 & 0 & 0 & i & 1-\lambda \end{vmatrix} = \lambda(\lambda-2)^2(\lambda-1)(\lambda+1) = 0$$

The solution to the above equation will give $\lambda = \{-1, 0, 1, 2, 2\}$ The normalized eigenvector corresponding to each eigenvalues are.

$$-1 \rightarrow {}^1\!/_{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad 0 \rightarrow {}^1\!/_{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \\ 1 \end{pmatrix} \quad 1 \rightarrow {}^1\!/_{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad 2 \rightarrow {}^1\!/_{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ 0 \end{pmatrix} \quad 2 \rightarrow {}^1\!/_{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -i \\ 1 \end{pmatrix}$$

So the transformation matrix to transform A to a diagonal matrix is

$$P = {}^1\!/_{\sqrt{2}} \begin{pmatrix} i & 0 & -i & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 & -i \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and its inverse is} \quad P^{-1} = {}^1\!/_{\sqrt{2}} \begin{pmatrix} -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 1 \\ i & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & i & 1 \end{pmatrix}$$

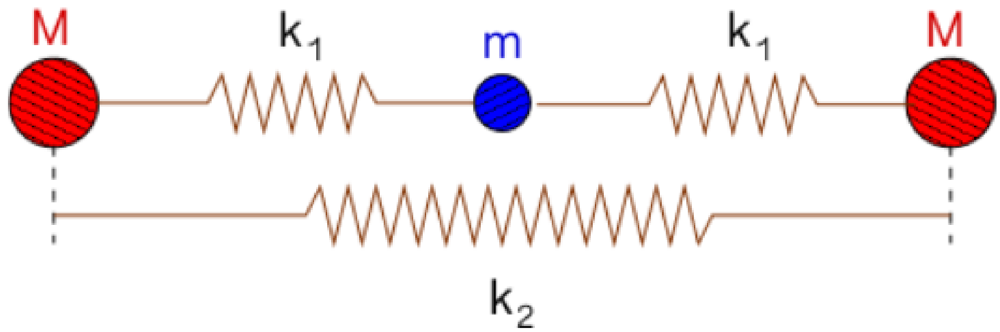
The diagonalization of A is done with $P^{-1}AP$ which is a diagonal matrix.

All matrices with real eigenvalues may not be hermition. Lets for example consider: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Its eigenvalue is 1 with multiplicity 2, which is real but the matrix is not Hermitian as:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

□

5. Find the normal modes and normal frequencies for linear vibrations (i.e. vibrations in the horizontal direction, as drawn) of the (over)simplified CO_2molecule modeled by the collection of masses and springs sketched below.



Solution:
If we suppose the displacement of each mass from their equilibrium position to be x_1, x_2 and x_3, then the kinetic energy of the system is the sum of kinetic energy of each masses which is:

$$T = \tfrac{1}{2}M\dot{x_1}^2 + \tfrac{1}{2}m\dot{x_2}^2 + \tfrac{1}{2}M\dot{x_3}^2$$

And the potential energy of the system would be:

$$V = \tfrac{1}{2}k_1(x_2 - x_1)^2 + \tfrac{1}{2}k_1(x_3 - x_2)^2 + \tfrac{1}{2}k_2(x_3 - x_1)$$

Now using the Lagranges equation of motion:

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{x_i}}\right) = \frac{\partial(T-V)}{\partial x_i}$$

Since T is free of x_is and V is free of $\dot{x_i}$ we can write the above expression as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x_i}}\right) = -\frac{\partial V}{\partial x_i}$$

calculating the above terms we get.

$$\begin{aligned}
M\ddot{x_1} &= -[(k_1+k_2)x_1 - k_1x_2 \quad - k_2x_3] \\
m\ddot{x_2} &= -[\quad -k_1x_1 + 2k_1x_2 \quad - k_1x_3] \\
M\ddot{x_3} &= -[\quad -k_2x_1 - k_1x_2 + (k_1+k_2)x_3]
\end{aligned}$$

The above set of relation can be written in matrix form as.

$$\begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix}\begin{pmatrix} \ddot{x_1} \\ \ddot{x_2} \\ \ddot{x_3} \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_1 & -k_2 \\ -k_1 & 2k_1 & -k_1 \\ -k_2 & -k_1 & k_1+k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

If we suppose that the motion is perfectly harmonic with frequency ω and suppose $x_k = \alpha e^{-i\omega x_k}$ Then $\ddot{x_i} = -\omega^2 x_i$. Using these values in above relation we get.

$$-\omega^2\begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_1 & -k_2 \\ -k_1 & 2k_1 & -k_1 \\ -k_2 & -k_1 & k_1+k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Writing above equation with matrix form as $(B-\omega^2 A)x = 0$, we can say that this equation has non trivial solutions for $|B-\omega^2 A| = 0$

$$\begin{vmatrix} -M\omega^2 + k_1 + k_2 & -k_1 & -k_2 \\ -k_1 & 2k_1 - m\omega^2 & -k_1 \\ -k_2 & -k_1 & -M\omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

$$\Rightarrow -2k_1^2 k_2 - 2k_1^2\left(-M\omega^2 + k_1 + k_2\right) - k_2^2\left(2k_1 - m\omega^2\right) + \left(2k_1 - m\omega^2\right)\left(-M\omega^2 + k_1 + k_2\right)^2 = 0$$

The solution for ω^2 for this equation are:

$$\left\{0, \quad \frac{1}{M}\left(k_1 + 2k_2\right), \quad \frac{1}{Mm}\left(2Mk_1 + k_1 m\right)\right\}$$

The first normal mode with $\omega^2 = 0$ implies that the system perform oscillation such that the relative position of the masses do not change meaning each mass oscillates in same direction with same frequency.

The second normal mode $\omega^2 = {}^1\!/_M(k_1 + 2k_2)$ doesn't depend upon the mass in the middle. So the middle mass remains at rest and the two mass at in the end perform oscillation with same frequency but opposite phase.

The third normal mode $\omega^2 = {}^1\!/_{Mm}\left(2Mk_1 + k_1 m\right)$ doesn't depend on the second spring with spring constant k_2 meaning the middle mass oscillates and the mass in the either remain at rest. □