All my Homeworks

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Preface

This document is the collection of the homework that I did during my time in Drexel University. You can find the individual homework for each course in my university webpage, which also contains link to some supporting documents like some python scripts I used and Cadabra2 stuffs I used for my tensor related works. This document is formatted in a book format where each chapter is a course and each section (named Homework One ... and so on) are the individual homemwork. If a problem is from the coursebook, then the problem number starts with the reference to the book and problem number in boldface within a pair of parenthesis.

As a caution, the solutions have a *lot* of errors. Honestly, after I got the graded homework back I have not made any serious attempt to look back my homework and correct them. I would say there is at least an average $\sim$10% overall mistake in the solutions. Apart from the incorrect solutions, there are few noticable errors and typos. There are some incomplete homework solutions too, which is partly because I did not have that last extra hour before the due date to typeset.

The way I typeset my homework has evolved considerably over the course of the period I did all these homeworks. I took some time to reformat/reorganize the old files but it this has not been complete significantly. Also, there are very obvious errors like some broken references and stuffs. So the problem in correcting those errors is not as trivial as it sounds sometimes. I usually organize my work in a hierarchial directory structure, and in compiling this one as a single `monolithic` document, I had to make sure that the relative `input` were correct. I got that part by a very clever trick, which I *might* share in a blog post, if I get in a mood to do so.

Then there is a problem of dublicate reference, which I am sure are plenty in document. This basically comes from the fact that when I originally did individual homework, I did not intend to compile them into a single document, and so the anchor labels were defined to be unique just within that one document, when I compiled them together there were a few that clashed, which I noticed and tried to correct. After the first time I compiled this big document, I started making sure that the individual homework would have that extra prefix so as to identify it uniquly, but I can't gurantee that it is error free in that regard too.

I would say most of what I have done here is completely my own work, apart from obvious inspiration I found in the internet off of other peoples work. I would say the one I have most influence from internet is my Electromagnetic theory II homework. One major part of the reason was that I did not put up as much as work in my coursework as I needed to, especially in the homework (I know what you are thinking at this point of time, let me tell you I have watched this video too). But still then most of them is my own work except when it is not. Since I did not plagiarize the thing, I have not given credit.

I owe a great deal of thank to my Professors for assigning these homework. Some of the unique homework problem they have desined have got me into serious thinking a lot of the time. Some times when I have been able to come up with the correct answer to these custom problem, it has given me as much joy as anything. I owe a huge thanks to my class mates, Andrew Antczak[1], Sean Lewis, Steve Sclafani, Wexiang[2] Yu, with whom I have had extensive discussion. Some of the works here are our collective work as a whole group or as pair or trio.

I would appreciate any comment or feedback. Any comment or feedback can be directed to pg459@drexel.edu. I might update this in the future.

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2019 Jun 12

[1] I am pretty sure I spelt his last name correct
[2] I am positive I spelt this one correct too

Contents

Chapter 1

Mathematical Physics

1.1 Homework One

1.1.1. Let a and b be any two vectors in a real linear vector space, and define $c = a + \lambda b$, where is real. By requiring that $c \geq 0$ for all λ, derive the Cauchy-Schwartz inequality

$$(a \cdot a)(b \cdot b) \geq (a, b)^2$$

When does equality hold? (Use only the general properties of the inner product. Do not assume that it is possible to write $a.b = |a||b| \cos(\theta)$
Solution:
Lets assume $c = a + \lambda b$, then by the definition of inner product we get $(c.c) \geq 0$

$$\begin{aligned}
c.c &= (a + \lambda b)(a + \lambda b) \geq 0 \\
&\Rightarrow ((a + \lambda b).a) + \lambda(a + \lambda b).b) \geq 0 && \text{Linearity of inner product} \\
&\Rightarrow (a.a) + \lambda(b.a) + \lambda(a.b) + \lambda^2(b.b) \geq 0 && \text{Linearity of inner product with } \lambda\text{real} \\
&\Rightarrow (a.a) + 2\lambda(a.b) + \lambda^2(b.b) \geq 0 && (a.b) = (b.a) \text{ for real vector space}
\end{aligned}$$

The above inequality is quadratic in λ which is real value. All the inner product map to real values in this real linear space. The equality of above expression is a quadratic equation of in r with real coefficient. Since the quadriatic equation lies wholely above real axis it can't have real solution. The condition for which is

$$\begin{aligned}
4(a.a)(b.b) &\leq (2(a.b))^2 \\
(a.a)(b.b) &\leq (a.b)^2.
\end{aligned}$$

Clearly the equality hold when the two vectors are identical. □

1.1.2. Let A be any square matrix, and define $B = e^A \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!}$ Prove that an eivenvector of A with eigenvalue λ is an eigenvector of B, with eigenvalue e^λ.
Solution:
Given λ is eigenvalue of A. Let the corresponding eivenvector be C. By defition

$$AC = \lambda C$$

Pre multiplying above relation with A we get.

$$A(AC) = A(\lambda C) \Rightarrow (AA)C = \lambda(AC) \Rightarrow A^2C = \lambda\lambda C = \lambda^2 C$$

By induction we get

$$A^nC = \lambda^n C \tag{1.1}$$

Now lets operate the vector C by B

$$\begin{aligned}
BC &= \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right) C && \text{(Definition of B Given)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} A^n C && \text{(Distributive property of Matrix over Matrix)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n C && \text{(From (1.1))} \\
&= \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) C && \text{(Distributive property of scalar over matrix)} \\
&= e^{\lambda} C && \text{(Definition of } e\text{)}
\end{aligned}$$

Since $BC = e^{\lambda}C$, C is the eigenvector of B with eigenvalue e^{λ}. □ □

1.1.3. Consider the 4-dimensional vector space of polynomials of degree less than or equal to 3, on the range $1 \leq x \leq 1$, spanned by the basis set $\left\{1, x, x^2, x^3\right\}$.The inner product of two polynomials in this space is defined as

$$(f, g) = \int_{-1}^{1} |x| f(x) g(x) dx$$

Use Gram–Schmidt orthogonalization to construct two orthonormal basis sets, as follows:

(a) Start with the set as listed above and begin the procedure with the function 1, as in class.

(b) Rewrite the set as $\{x^2, x, 1, x^3\}$and begin the orthogonalization procedure starting with x^2

Write down the matrix representing the transformation from basis (i) to basis (ii), and demon- strate that it is orthogonal.

Solution:

Let us write the cross product table for the given basis functions.

$$(1 \cdot 1) = \int_{-1}^{1} |x| 1 \cdot 1 dx = \int_{-1}^{0} -x dx + \int_{0}^{1} x dx = \left.\frac{-x^2}{2}\right]_{-1}^{0} + \left.\frac{-x^2}{2}\right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

$$(1 \cdot x) = \int_{-1}^{1} |x| 1 \cdot x dx = \int_{-1}^{0} -x^2 dx + \int_{0}^{1} x^2 dx = \left.\frac{-x^3}{3}\right]_{-1}^{0} + \left.\frac{x^3}{3}\right]_{0}^{1} = -\frac{1}{3} + \frac{1}{3} = 0$$

$$(1 \cdot x^2) = \int_{-1}^{1} |x| 1 \cdot x^2 dx = \int_{-1}^{0} -x^3 dx + \int_{0}^{1} x^3 dx = \left.\frac{-x^4}{4}\right]_{-1}^{0} + \left.\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$(1 \cdot x^3) = \int_{-1}^{1} |x| 1 \cdot x^3 dx = \int_{-1}^{0} -x^4 dx + \int_{0}^{1} x^4 dx = \left.\frac{-x^5}{5}\right]_{-1}^{0} + \left.\frac{x^5}{5}\right]_{0}^{1} = -\frac{1}{5} + \frac{1}{5} = 0$$

$$(x \cdot 1) = (1 \cdot x) = 0$$

$$(x \cdot x) = \int_{-1}^{1} |x| x \cdot x dx = \int_{-1}^{1} |x| 1 \cdot x^2 dx = \frac{1}{2}$$

$$(x \cdot x^2) = \int_{-1}^{1} |x| x \cdot x^2 dx = \int_{-1}^{1} |x| 1 \cdot x^3 dx = 0$$

$$(x \cdot x^3) = \int_{-1}^{1} |x| x \cdot x^3 dx = \int_{-1}^{0} -x^5 dx + \int_{0}^{1} x^5 dx = \left.\frac{-x^6}{6}\right]_{-1}^{0} + \left.\frac{x^6}{6}\right]_{0}^{1} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$(x^2 \cdot 1) = (1 \cdot x^2) = \frac{1}{2} \qquad (x^2 \cdot x) = (x \cdot x^2) = 0$$

$$(x^2 \cdot x^2) = \int_{-1}^{1} |x| x^2 \cdot x^2 dx = \int_{-1}^{1} |x| x \cdot x^3 dx = \frac{1}{3}$$

$$(x^2 \cdot x^3) = \int_{-1}^{1} |x| x^2 \cdot x^3 dx = \int_{-1}^{0} -x^6 dx + \int_{0}^{1} x^6 dx = \left.\frac{-x^7}{7}\right]_{-1}^{0} + \left.\frac{x^7}{7}\right]_{0}^{1} = -\frac{1}{7} + \frac{1}{7} = 0$$

$$(x^3 \cdot 1) = (1 \cdot x^3) = 0 \qquad (x^3 \cdot x) = (x \cdot x^3) = \frac{1}{2} \qquad (x^3 \cdot x^2) = (x^3 \cdot x^2) = 0$$

$$(x^3 \cdot x^3) = \int_{-1}^{1} |x| x^3 \cdot x^3 dx = \int_{-1}^{0} -x^7 dx + \int_{0}^{1} x^7 dx = \left.\frac{-x^8}{8}\right]_{-1}^{0} + \left.\frac{x^8}{8}\right]_{0}^{1} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

For the first basis set: $1, x, x^2, x^3$; Let $(v_1, v_2, v_3, v_4) = (1, x, x^2, x^3)$ and let $(\hat{e_1}, \hat{e_4}, \hat{e_3}, \hat{e_4})$ be the corresponding orthonormal set.

$$e_1 = v_1 = 1 \qquad \hat{e_1} = \frac{e_1}{\sqrt{(e_1, e_1)}} = \frac{1}{\sqrt{(1 \cdot 1)}} = 1$$

$$e_2 = v_2 - (\hat{e_1}, v_2)\hat{e_1} \qquad \hat{e_2} = \frac{x}{\sqrt{(x, x)}} = \frac{1}{\sqrt{1/2}} = \sqrt{2}x$$

$$= x - (1, x)1 = x - 0 = x$$

The normal vector corresponding to v_3 can be found as.

$$
\begin{aligned}
e_3 &= v_3 - (\hat{e_2}, v_3)\hat{e_2} - (\hat{e_1}, v_3)\hat{e_1} \\
&= x^2 - (\sqrt{2}x, x^2)\sqrt{2}x - (1, x^2)1 \\
&= x^2 - \sqrt{2}(x, x^2)\sqrt{2}x - \frac{1}{2} \\
&= x^2 - \sqrt{2}\cdot 0 \cdot \sqrt{2}x - \frac{1}{2} \\
&= x^2 - \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\hat{e_3} &= \frac{x^2 - 1/2}{\sqrt{(x^2 - 1/2, x^2 - 1/2)}} \\
&= \frac{x^2 - 1/2}{\sqrt{(x^2, x^2) - 2(x^2, 1/2) + (1/2, 1/2)}} \\
&= \frac{x^2 - 1/2}{\sqrt{\frac{1}{3} - 2\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}}} \\
&= \sqrt{3}\left(2x^2 - 1\right)
\end{aligned}
$$

For $v_4 = x^2$ we similarly get:

$$
\begin{aligned}
e_4 &= v_4 - (\hat{e_3}, v_4)\hat{e_3} - (\hat{e_2}, v_4)\hat{e_2} - (\hat{e_1}, v_4)\hat{e_1} \\
&= x^3 - (\sqrt{3}\left(2x^2 - 1\right), x^3)\sqrt{3}\left(2x^2 - 1\right) - (\sqrt{2}x, x^3)\sqrt{2}x - (1, x^3)1 \\
&= x^3 - \sqrt{3}\left(2(x^2, x^3) - (1, x^3)\right)\sqrt{3}\left(2x^2 - 1\right) - \sqrt{2}(x, x^3)\sqrt{2}x \\
&= x^3 - \sqrt{3}\left(0 - 0\right)\sqrt{3}\left(2x^2 - 1\right) - \sqrt{2}\frac{1}{3}\sqrt{2}x \\
&= x^3 - 0 - \frac{2}{3}x \\
&= x^3 - \frac{2}{3}x
\end{aligned}
$$

$$
\begin{aligned}
\hat{e_3} &= \frac{x^3 - (2/3)x}{\sqrt{\left(x^3 - \frac{2}{3}x, x^3 - \frac{2}{3}x\right)}} \\
&= \frac{x^3 - (2/3)x}{\sqrt{\left((x^3, x^3) - 2\frac{2}{3}(x^3, x) - \frac{2}{3}^2(x, x)\right)}} \\
&= \frac{x^3 - (2/3)x}{\sqrt{\frac{1}{4} - 2\cdot\frac{2}{3}\cdot 0 + \frac{2}{9}}} \\
&= 6\left(x^3 - \frac{2}{3}x\right) = 6x^3 - 4x
\end{aligned}
$$

Therefore the orthonormalized basis set is

$$
\left\{1, \quad \sqrt{2}x, \quad \sqrt{3}(2x^2 - 1), \quad 6x^3 - 4x\right\}
$$

Working out similarly,

Let $(\hat{e_1}', \hat{e_4}', \hat{e_3}', \hat{e_4}')$ be the corresponding orthonormal set.

$$
e_1' = v_1 = x^2
$$

$$
\hat{e_1}' = \frac{e_1'}{\sqrt{(e_1', e_1')}} = \frac{x^2}{\sqrt{(x^2, x^2)}} = \frac{x^2}{\sqrt{1/3}} = \sqrt{3}x^2
$$

$$
\begin{aligned}
e_2' &= v_2 - (\hat{e_1}', v_2) \\
&= x - (\sqrt{3}x^2, x)\sqrt{3}x^2 \\
&= x - \sqrt{3}\cdot 0 \cdot \sqrt{3}x^2 = x
\end{aligned}
$$

$$
\hat{e_2}' = \frac{e_2'}{\sqrt{(e_2', e_2')}} = \frac{x}{\sqrt{(x, x)}} = \frac{1}{\sqrt{1/2}} = \sqrt{2}x
$$

$$
\begin{aligned}
e_3' &= v_3 - (\hat{e_2}', v_3)\hat{e_2}' - (\hat{e_1}', v_3)\hat{e_1}' \\
&= 1 - (\sqrt{2}x, 1)\sqrt{2}x - (\sqrt{3}x^2, 1)\sqrt{3}x^2 \\
&= 1 - \sqrt{2}(x, 1)\sqrt{2}x - \sqrt{3}(x^2, 1)\sqrt{3}x^2 \\
&= 1 - \sqrt{2}\cdot 0 \cdot \sqrt{2}x - \sqrt{3}\cdot\frac{1}{2}\cdot\sqrt{3}x^2 \\
&= 1 - \frac{3}{2}x^2
\end{aligned}
$$

$$
\begin{aligned}
(e_3', e_3') &= \left(1 - {}^3\!/_2 x^2, 1 - {}^3\!/_2 x^2\right) \\
&= (1, 1) - 2(1, -{}^3\!/_2 x^2) + (-{}^3\!/_2 x^2, -{}^3\!/_2 x^2) \\
&= 1 + 2\,{}^3\!/_2(1, x^2) + ({}^3\!/_2)^2(x^2, x^2) \\
&= 1 + 2\cdot {}^3\!/_2 \cdot {}^1\!/_2 + ({}^3\!/_2)^2 0 = 4 \\
\hat{e_3}' &= \frac{e_3'}{\sqrt{(e_3', e_3')}} = \frac{1 - {}^3\!/_2 x^2}{\sqrt{4}} = 2 - 3x^2
\end{aligned}
$$

$$
\begin{aligned}
e_4' &= v_4 - (\hat{e_3}', v_4)\hat{e_3}' - (\hat{e_2}', v_4)\hat{e_2}' - (\hat{e_1}', v_4)\hat{e_1}' \\
&= x^3 - (2 - 3x^2, x^3)(2 - 3x^2) - (\sqrt{2}x, x^3)\sqrt{2}x - (\sqrt{3}x^2, x^3)\sqrt{3}x^2 \\
&= x^3 - (2(1, x^3) - 3(x^2, x^3))(2 - 3x^2)) - \sqrt{2}(x, x^3)\sqrt{2}x - \sqrt{3}(x^2, x^3)\sqrt{3}x^2 \\
&= x^3 - (2 \cdot 0 - 3 \cdot 0)(2 - 3x^2)) - \sqrt{2}({}^1\!/_3)\sqrt{2}x - \sqrt{3}(0)\sqrt{3}x^2 \\
&= x^3 - \sqrt{2}({}^1\!/_3)\sqrt{2}x = x^3 - \frac{2}{3}x
\end{aligned}
$$

$$
\begin{aligned}
(e_4', e_4') &= \left(x^3 - {}^2\!/_3 x, x^3 - {}^2\!/_3 x\right) \\
&= (x^3, x^3) - 2\,{}^2\!/_3(x^3, x) + ({}^2\!/_3)^2(x, x \\
&= {}^1\!/_4 - 2 \cdot {}^2\!/_3 \cdot 0 + {}^2\!/_9 \\
&= {}^1\!/_{36} \\
\hat{e_4}' &= \frac{e_4}{\sqrt{(e_4, e_4)}} = \frac{x^3 - {}^2\!/_3 x}{\sqrt{{}^1\!/_{36}}} = 6x^3 - 4
\end{aligned}
$$

So the orthonormal basis set is found to be.

$$
\left\{ \sqrt{3}x^2, \quad \sqrt{2}x, \quad -3x^2 + 2, \quad 6x^3 - 4x \right\}
$$

We can now find the transformation matrix between these two basis set. If we suppose λ_{ij} be the elements of the transformation. Then $\hat{e}\ \lambda_{ij} = \left(\hat{e_i}, \hat{e_j}'\right)$ So,

$$
\begin{aligned}
\lambda_{11} &= \left(\hat{e_1}, \hat{e_1}'\right) = \left(1, \sqrt{3}x^2\right) = \sqrt{3}(1, x^2) = \sqrt{3}\,{}^1\!/_2 \\
\lambda_{12} &= \left(\hat{e_1}, \hat{e_2}'\right) = \left(1, \sqrt{2}x\right) = \sqrt{2}(1, x) = 0 \\
\lambda_{13} &= \left(\hat{e_1}, \hat{e_3}'\right) = \left(1, -3x^2 + 2\right) = -3(1, x^2) + 2(1, 1) = -3\,{}^1\!/_2 + 2 = {}^1\!/_2 \\
\lambda_{14} &= \left(\hat{e_1}, \hat{e_4}'\right) = \left(1, 6x^3 - 4x\right) = 6(1, x^3) + 4(1, x) = 6 \cdot 0 + 4 \cdot 0 = 0 \\
\lambda_{21} &= \left(\hat{e_2}, \hat{e_1}'\right) = \left(\sqrt{2}x, \sqrt{3}x^2\right) = \sqrt{2 \times 3}(x, x^2) = 0 \\
\lambda_{22} &= \left(\hat{e_2}, \hat{e_2}'\right) = \left(\sqrt{2}x, \sqrt{2}x\right) = \sqrt{2 \times 2}(x, x) = 2\,{}^1\!/_2 = 1 \\
\lambda_{23} &= \left(\hat{e_2}, \hat{e_3}'\right) = \left(\sqrt{2}x, -3x^2 + 2\right) = -3\sqrt{2}(x, x^2) + \sqrt{2}(x, 1) = -3\sqrt{2} \cdot 0 + \sqrt{2} \cdot 0 = 0 \\
\lambda_{24} &= \left(\hat{e_2}, \hat{e_4}'\right) = \left(\sqrt{2}x, 6x^3 - 4x\right) = 6\sqrt{2}(1, x^3) + 4\sqrt{2}(1, x) = 6\sqrt{2} \cdot 0 + 2\sqrt{2} \cdot 0 = 0
\end{aligned}
$$

Wroking this out we get the transformation matrix as.

$$
\begin{pmatrix}
\sqrt{3}\,{}^1\!/_2 & 0 & {}^1\!/_2 & 0 \\
0 & 1 & 0 & 0 \\
{}^1\!/_2 & 0 & -\sqrt{3}\,{}^1\!/_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

For an orthogonal matrix A, the inverse $A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. To prove that the matrix is orthogonal it suffices to show that the product of the matrix and its transpose is identity matrix.

$$
\begin{pmatrix}
\sqrt{3}\,{}^1\!/_2 & 0 & {}^1\!/_2 & 0 \\
0 & 1 & 0 & 0 \\
{}^1\!/_2 & 0 & -\sqrt{3}\,{}^1\!/_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
\sqrt{3}\,{}^1\!/_2 & 0 & {}^1\!/_2 & 0 \\
0 & 1 & 0 & 0 \\
{}^1\!/_2 & 0 & -\sqrt{3}\,{}^1\!/_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

Since $AA^T = I, A^T$ is the inverse of matrix A which shows that the matrix A is orthogonal. □

1.1.4. (a) Transform the matrix A into a coordinate system in which A is diagonal, with the diagonal elements increasing from top to bottom. Write down the transformation matrix and the diagonalized A

$$A = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & i & 1 \end{pmatrix}$$

(b) A matrix B has real eigenvalues. Does it necessarily follow that B is hermitian? Prove the statement or give a counterexample.

Solution:

Lets find the eigenvalues of the matrix A. The determinant of $(A - \lambda I)$ is

$$\begin{vmatrix} -\lambda & -i & 0 & 0 & 0 \\ i & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & -i \\ 0 & 0 & 0 & i & 1-\lambda \end{vmatrix} = \lambda(\lambda-2)^2(\lambda-1)(\lambda+1) = 0$$

The solution to the above equation will give $\lambda = \{-1, 0, 1, 2, 2\}$ The normalized eigenvector corresponding to each eigenvalues are.

$$-1 \to {}^1\!/\!_{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad 0 \to {}^1\!/\!_{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \\ 1 \end{pmatrix} \quad 1 \to {}^1\!/\!_{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad 2 \to {}^1\!/\!_{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ 0 \end{pmatrix} \quad 2 \to {}^1\!/\!_{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -i \\ 1 \end{pmatrix}$$

So the transformation matrix to transform A to a diagonal matrix is

$$P = {}^1\!/\!_{\sqrt{2}} \begin{pmatrix} i & 0 & -i & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 & -i \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and its inverse is} \quad P^{-1} = {}^1\!/\!_{\sqrt{2}} \begin{pmatrix} -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 1 \\ i & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & i & 1 \end{pmatrix}$$

The diagonalization of A is done with $P^{-1}AP$ which is a diagonal matrix.

All matrices with real eigenvalues may not be hermition. Lets for example consider: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Its eigenvalue is 1 with multiplicity 2, which is real but the matrix is not Hermitian as:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

□

1.1.5. Find the normal modes and normal frequencies for linear vibrations (i.e. vibrations in the horizontal direction, as drawn) of the (over)simplified “CO_2molecule” modeled by the collection of masses and springs sketched below.

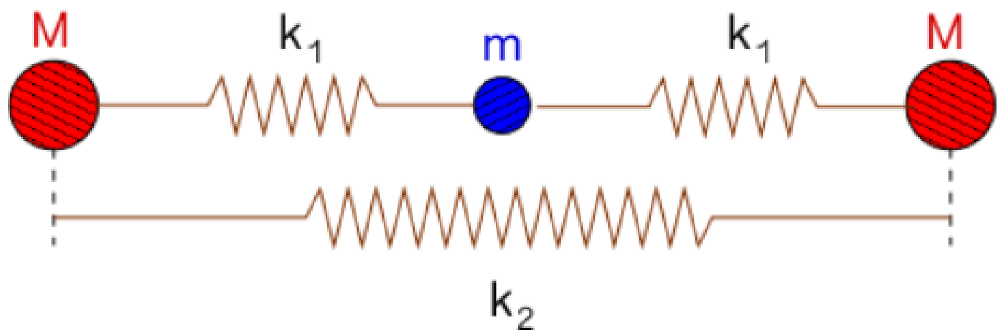
Solution:

If we suppose the displacement of each mass from their equilibrium position to be x_1, x_2 and x_3, then the kinetic energy of the system is the sum of kinetic energy of each masses which is:

$$T = {}^1\!/\!_2 M\dot{x_1}^2 + {}^1\!/\!_2 m\dot{x_2}^2 + {}^1\!/\!_2 M\dot{x_3}^2$$

And the potential energy of the system would be:

$$V = {}^1\!/\!_2 k_1(x_2 - x_1)^2 + {}^1\!/\!_2 k_1(x_3 - x_2)^2 + {}^1\!/\!_2 k_2(x_3 - x_1)$$



Now using the Lagranges equation of motion:

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{x}_i}\right) = \frac{\partial(T-V)}{\partial x_i}$$

Since T is free of x_is and V is free of $\dot{x}_i$ we can write the above expression as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = -\frac{\partial V}{\partial x_i}$$

calculating the above terms we get.

$$\begin{aligned} M\ddot{x}_1 &= -[(k_1+k_2)x_1 - k_1x_2 \qquad - k_2x_3] \\ m\ddot{x}_2 &= -[\qquad -k_1x_1 + 2k_1x_2 \qquad - k_1x_3] \\ M\ddot{x}_3 &= -[\qquad -k_2x_1 - k_1x_2 + (k_1+k_2)x_3] \end{aligned}$$

The above set of relation can be written in matrix form as.

$$\begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_1 & -k_2 \\ -k_1 & 2k_1 & -k_1 \\ -k_2 & -k_1 & k_1+k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

If we suppose that the motion is perfectly harmonic with frequency ω and suppose $x_k = \alpha e^{-i\omega x_k}$ Then $\ddot{x}_i = -\omega^2 x_i$. Using these values in above relation we get.

$$-\omega^2\begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_1 & -k_2 \\ -k_1 & 2k_1 & -k_1 \\ -k_2 & -k_1 & k_1+k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Writing above equation with matrix form as $(B-\omega^2 A)x = 0$, we can say that this equation has non trivial solutions for $|B-\omega^2 A| = 0$

$$\begin{vmatrix} -M\omega^2+k_1+k_2 & -k_1 & -k_2 \\ -k_1 & 2k_1-m\omega^2 & -k_1 \\ -k_2 & -k_1 & -M\omega^2+k_1+k_2 \end{vmatrix} = 0$$

$$\Rightarrow -2k_1^2k_2 - 2k_1^2\left(-M\omega^2+k_1+k_2\right) - k_2^2\left(2k_1-m\omega^2\right) + \left(2k_1-m\omega^2\right)\left(-M\omega^2+k_1+k_2\right)^2 = 0$$

The solution for ω^2 for this equation are:

$$\left\{0, \quad \frac{1}{M}\left(k_1+2k_2\right), \quad \frac{1}{Mm}\left(2Mk_1+k_1m\right)\right\}$$

The first normal mode with $\omega^2 = 0$ implies that the system perform oscillation such that the relative position of the masses do not change meaning each mass oscillates in same direction with same frequency.

The second normal mode $\omega^2 = {}^1\!/_M(k_1 + 2k_2)$ doesn't depend upon the mass in the middle. So the middle mass remains at rest and the two mass at in the end perform oscillation with same frequency but opposite phase.

The third normal mode $\omega^2 = {}^1\!/_{Mm}\,(2Mk_1 + k_1 m)$ doesn't depend on the second spring with spring constant k_2 meaning the middle mass oscillates and the mass in the either remain at rest. □

1.2 Homework Two

1.2.1. An electrical network consists of N interconnected nodes. Each pair of nodes (i, j) is connected by a resistor of resistance $R_{ij} = \min(i, j) + 2\max(i, j)$, for $i, j = 1, ..., N$. Let V_i be the electrical potential of node i, and choose the zero level of potential to set $V_1 = 0$. Then Kirchhoff's laws for the other nodes in the network can be conveniently written as

$$\sum_{j=1}^{N} \frac{V_j - V_i}{R_{ij}} = I_i$$

, for $i = 2, ..., N$ where I_i is the current flowing from node i to some external circuit. Suppose N = 100 and the external connection is such that current flows out of node 2 and back into node 1, so $I_1 = 1$, $I_2 = 1$, and $I_i = 0$ for $i > 2$. By solving the above $(N1)$-dimensional matrix equation, calculate the total resistance between nodes 1 and 2.

Solution:

Expanding the i^{th} current value

$$\begin{aligned} I_1 &= \frac{V_2 - V_1}{R_{12}} + \frac{V_3 - V_1}{R_{13}} + \cdots + \frac{V_N - V_1}{R_{1N}} \\ &= -\left(\frac{1}{R_{12}} + \frac{1}{R_{12}} + \cdots + \frac{1}{R_{1N}}\right) V_1 + \frac{V_2}{R_{12}} + \frac{V_3}{R_{13}} + \cdots + \frac{V_N}{R_{1N}} \end{aligned}$$

Smililarly expanding all others we get the pattern. So the equivalent matrix would be.

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{R_{12}} + \frac{1}{R_{12}} + \cdots + \frac{1}{R_{1N}}\right) & \frac{1}{R_{12}} & \frac{1}{R_{13}} & \cdots & \frac{1}{R_{1N}} \\ \frac{1}{R_{21}} & -\left(\frac{1}{R_{21}} + \frac{1}{R_{23}} + \cdots + \frac{1}{R_{2N}}\right) & \frac{1}{R_{23}} & \cdots & \frac{1}{R_{2N}} \\ \vdots & \cdots & \ddots & & \vdots \\ \frac{1}{R_{N1}} & \frac{1}{R_{N2}} & \frac{1}{R_{N3}} & \cdots & -\left(\frac{1}{R_{N1}} + \frac{1}{R_{N2}} + \cdots + \frac{1}{R_{NN}}\right) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{pmatrix}$$

This is the Matrix equation relating the Ohm's law where $I = \frac{V}{R}$. The $1/R$ Matrix is $N \times N$ matrix. But since V_1 is zero and I_1 is known we can eleminate the first row and column of $1/R$ Matrix to get $(N-1)$ Dimensional Matrix

```python
import numpy as np
from numpy import linalg as LA

num_nodes = 100
#Construct the matrix
N = num_nodes
def Res(i,j):
    return min(i,j) + 2*max(i,j)

# Construct the R matrix with every elements except the diagonals
# int(i!=j) returns 1 for non diagonal places and 0 for diagonal
# places so all the diagonal elements
R=np.array([(int(i!=j)*(1/Res(i,j))) for i in range(1,N+1) for j in range(1,N+1)]).reshape(N,
    N)

# Since  in the matrix we see that the  diagonal
# elements are  simply negative sum of all other
# elements in the matrix, we sum them to get the
# diagonal elements and  put them back to matrix
Diag = [-sum(R[i]) for i in range(N)]
np.fill_diagonal(R,Diag)

#Deleting the first row and colums
R = R[1:,1:]

#Initializing the Current column matrix to zero
I = np.zeros(N-1).reshape(N-1,1)

# The first value of current I_2 is indexed at zero
# so I[0] means the I_2 which is 1
I[0] = 1

# Solving the matrix equation to get the potentials at all nodes
V =LA.solve(R,I).reshape(N-1)

# Since our zero index in program is index 2 for current and voltages
# the voltage at node 2 is V[0]
V2 = V[0]
print('The  electric potential  at  node 2 is {:.4}V'.format(V[0]))

# The current in the whole circuit is 1A so the equivalent resistance
# can be calculated by Ohm's law. R_eq = (V1 - V2) / I, since V1 = 0
# The total resistance of circuit is just R_eq  = -V2/1 = -V2
print('The equivalent Resistance of circuit is Ω{:.4}'.format(-V[0]))
```

```
The  electric potential at   node 2 is -0.9442V
The equivalent Resistance of circuit is 0.9442Ω
```

□

1.2.2. The data file hw2.2.dat on the course Web page contains (hypothetical) experimental data on the measurement of a function y(x). The N data points are arranged, one measurement per line, in the format

xi   yi (measured) σi

where σ_i is an estimate of the uncertainty in the i-th measurement. It is desired to find the least-square fit to the data by polynomials of the form

$$y(x) = \sum_{j=1}^{N} a_j x^{j-1}$$

for specified values of m, by minimizing the quantity

$$\chi^2 = \sum_{i=1}^{N} \left[\frac{y_i - \sum_{j=1}^{m} a_j x^{j-1}}{\sigma_i} \right]^2$$

As discussed in class (and in Numerical Recipes, pp 671–676), write down the overdetermined design matrix equation that results from writing $y(x_i) = y_i$,

$$Aa = b$$

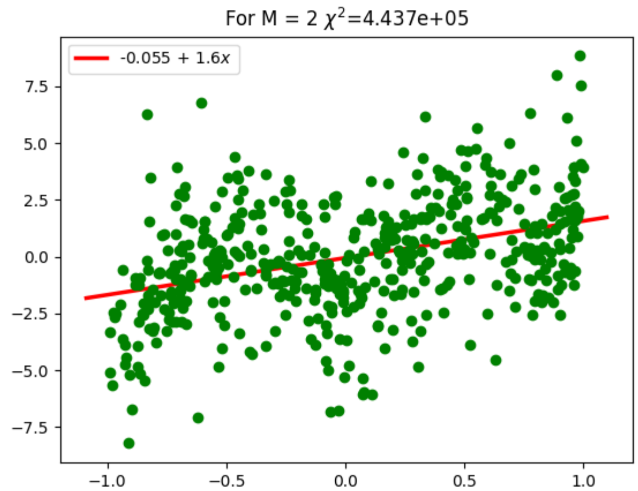
, where $A_{ij} = \frac{x_i^{j1}}{\sigma_i}$, $b_i = \frac{y_i}{\sigma_i}$ (so the measurement undertainties are included in each row), and a is the vector of unknown coefficients. Solve this system using singular value decomposition (svdcmp in Numerical Recipes, svd in Python or Matlab) to obtain the best fitting polynomial for each of the cases m = 2, 4, 7, and 13. For each m, give the values of a_j and χ^2, and plot the data and the best fit on a single graph.

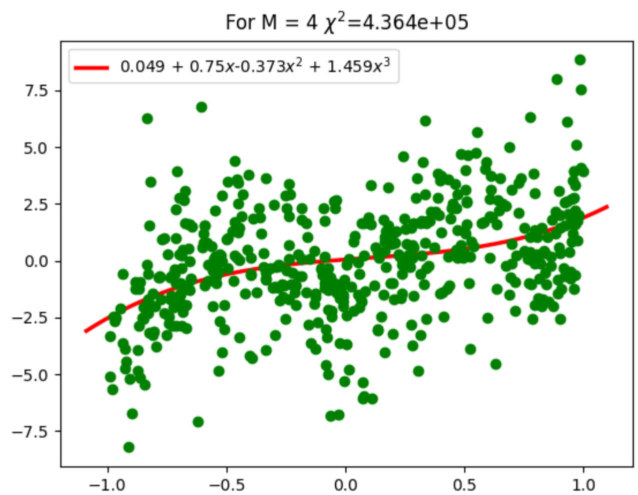
Solution:

```
#!/usr/bin/env python3

import numpy as np
import numpy.linalg as LA
import matplotlib.pyplot as plt
from matplotlib2tikz import save as tikz_save

class LeastSquare():
    mlist = [2,4,7,13]#6,7,8,9,13]
    pltcnt = len(mlist)
    pltprm = 221
    clr = 0.1 #clearence

    datafile = './data/hw2.2.dat'
    slc = 500  # slice length to test for fewer data points
    epsilon = 1e-3 # zero threshold for svd inverting

    def __init__(self):
        self.readfile()

    def readfile(self):
        read = np.genfromtxt(self.datafile)
        self.x  = read[:,0]; self.x = self.x[:self.slc]
        self.y  = read[:,1]; self.y  = self.y[:self.slc]
        self.sd = read[:,2]; self.sd = self.sd[:self.slc]
        self.err  = self.sd.reshape(self.slc,1)
        return read

    def construct_A(self,M):
        #Reform x shape to column shape
        xcol = self.x.reshape(self.slc,1)
        # Same for error values
        err  = self.sd.reshape(self.slc,1)

        # initialize the first column of the A matrix
        A = (np.zeros(self.slc) + 1).reshape(self.slc,1)/err
        for m in range(1,M):
            A = np.append(A,xcol**m/err,1)

        return np.matrix(A)

    def construct_b(self):
        coly = self.y.reshape(len(self.y),1)
        return np.matrix(coly/self.err)

    def get_svd_inverse(self,M):
```

```
        U,W,V = LA.svd(M)
        WI_star = []
        for wi in W:
            if wi < self.epsilon:
                WI_star.append(0)
            else:
                WI_star.append(1/wi)
        WI = np.diag(WI_star)
        r,c = M.shape
        inc_prm = r-c
        cm = np.matrix(np.zeros(inc_prm*c).reshape(c,inc_prm))
        WI = np.hstack([WI,cm])
        return V.T*WI*U.T

    def get_polynomial_values(self,aj,x):
        y = np.zeros(len(x))
        for j in range(len(aj)):
            y += aj[j]*x**j #since our index starts at 0, it works
        return y

    def get_coefficient(self,m):
        A = self.construct_A(m)
        b = self.construct_b()
        AI = self.get_svd_inverse(A)
        a = AI*b
        #print('a shape is {}'.format(a.shape))
        ar = np.squeeze(np.asarray(a))
        return ar

    def get_chi_sq(self,true,predicted,error):
        return np.sum(((true-predicted)/error)**2)

    def get_legend(self,aj):
        j = 2
        lg = r'{:.2} + {:.2}$x$'.format(aj[0],aj[1])
        for a in aj[j:]:
            sgn = ' + '
            if a < 0: sgn = ''
            t = sgn + r'{:.4}$x^'.format(a)+'{'+'{}'.format(j)+'}$'
            lg += t
            j += 1

        return lg



    def draw_graph(self):
        cnt = 0
        for m in self.mlist:
            #plt.subplot(self.pltprm+cnt); cnt+=1

            aj = self.get_coefficient(m)
            #print('ar was ',ar)
            x = np.linspace(1.1*min(self.x),1.1*max(self.x),500)
            y = self.get_polynomial_values(aj,x)

            prediction = self.get_polynomial_values(aj,self.x)

            chisq = self.get_chi_sq(self.y,prediction,self.err)

            plt.plot(x,y,'r',linewidth='2.5',label=self.get_legend(aj))
            plt.legend()
            plt.plot(self.x,self.y,'go')
            plt.title('For M = {} $\chi^2$={:.3e}'.format(m,chisq))
            plt.savefig('M{}.png'.format(m))
            plt.show()

        plt.suptitle('M degree polynomial least square fit by SVD')
```

```
        #tikz_save('Least square plots.tex',figurewidth="14cm",figureheight="9cm")
        plt.show()


if __name__ == '__main__':
    LS = LeastSquare()
    LS.draw_graph()
```



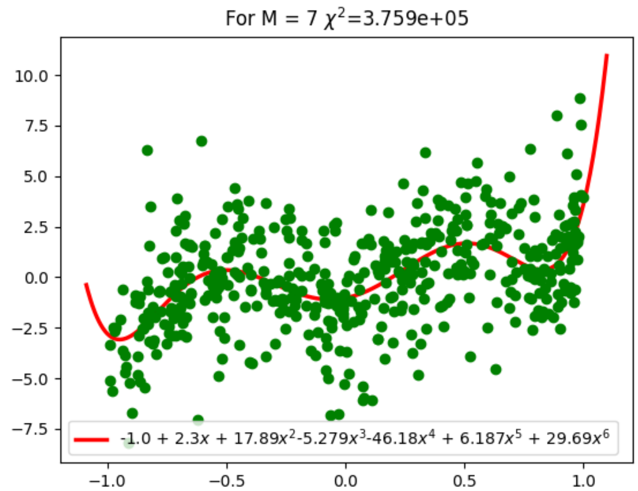


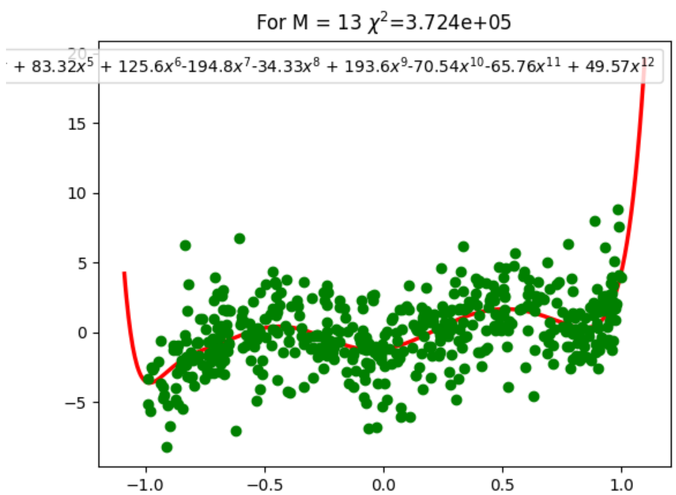
The a_j values are shown as polynomial coefficient in the legend of each plot and the χ^2 values are given at the top of each graph. In each case the continuous line is the fitted polynomial and the scattered dots are the values read from file. □

1.2.3.

Solution:
This Question was mostly solved by use of Sympy package in python.

```
import numpy as np
import numpy.linalg as LA
```






```python
from scipy.integrate import quad

import sympy as smp
import sympy.functions as smf
import sympy.physics.quantum.constants as qc
```

The function $\phi_n(x)$ was defined to get any order function

```python
def fai(self,n):
    x = self.x; b = self.b
    of = smp.Rational(1,4)
    oh = smp.Rational(1,2)
    A = (b**2/smp.pi)**of*1/(smp.sqrt(2**n*smf.factorial(n)))
    return (smp.exp(-oh*b**2*x**2)*smf.hermite(n,b*x)) * A
```

So for example this function gives for $fai(n)$ as.

$$\frac{2^{-\frac{p}{2}}\sqrt{|\beta|}}{\sqrt[4]{\pi}\sqrt{p!}}e^{-\frac{\beta^2x^2}{2}}H_p(\beta x)$$

We wish to approximate the energy eigenfunctions of a one-dimensional square well by expanding them in terms of a *finite* (N-dimensional) subset of harmonic oscillator wavefunctions. The square well is defined by the potential

$$V(x) = \begin{cases} 0 & (|x| < a)\,, \\ V_0 & (|x| > a)\,. \end{cases}$$

The harmonic oscillator potential is $V_{ho}(x) = \frac{1}{2}kx^2$, where we will take $k = 2V_0/a^2$ here. As discussed in class, solving the problem entails diagonalization (e.g. using the Python function `eigvals` or the *Numerical Recipes* functions `tred2` and `tqli`) of the Hamiltonian matrix $H = (h_{nm})$, where

$$h_{nm} = \langle n|H|m\rangle = \int dx\, \phi_n^*(x) \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right] \phi_m(x)$$

and $\phi_n(x)$ is the n-th harmonic oscillator wavefunction:

$$\phi_n(x) = \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}\, e^{-\frac{1}{2}\beta^2 x^2} H_n(\beta x)\,,$$

with $\beta^4 = mk/\hbar^2$.

Use the recurrence relations given in Riley & Hobson, p. 373, to generate the H_n, and the differential relations (same page) along with the trapezoidal rule, where needed, to compute the matrix elements h_{nm}.

Hence, by diagonalizing the matrix H, determine the first (and only) two energy levels E_0 and E_1 of a square well with $V_0 a^2 = 2\hbar^2/m$, for three different values of N: (a) use $\phi_0, \ldots, \phi_4$ as a basis ($N = 5$); (b) use $\phi_0, \ldots, \phi_9$ ($N = 10$); and (c) use $\phi_0, \ldots, \phi_{19}$ ($N = 20$). In each case, compare your answers with the exact values

$$E_0 = 0.53\frac{\hbar^2}{ma^2}\,, \quad E_1 = 1.80\frac{\hbar^2}{ma^2}\,.$$

Then the hamiltonial operator function is defined as.

```
def H(self,psi):
   hf = smp.Rational(1,2)
   x = self.x; m = self.m; hcut = self.hcut
   hctm = -hf*(hcut**2)/(m)
   return hctm * smp.diff(psi,x,2) + self.V(x)*psi
```

Writing $A_p = \frac{2^{-\frac{p}{2}}\sqrt{|\beta|}}{\sqrt[4]{\pi}\sqrt{p!}}$ and after operating $\phi_p(x)$ by Hamiltonian operator and using the recurrence relation we get.

$$\hat{H}\phi_p(x) = \frac{A_p e^{-\frac{\beta^2 x^2}{2}}}{2m}\left(-\beta^2\hbar^2\left(\beta^2 x^2 H_p(\beta x) - 4\beta p x H_{p-1}(\beta x) + 4p(p-1)H_{p-2}(\beta x) - H_p(\beta x)\right) + kmx^2 H_p(\beta x)\right)$$

If we evaluate the functions operated by at numeric value 0.

$$\hat{H}\phi_0(x) = \frac{A_p e^{-\frac{\beta^2 x^2}{2}}}{2m}\left(-\beta^2\hbar^2\left(\beta^2 x^2 - 1\right) + kmx^2\right)$$

Evaluating function $\phi_0(x)$ we get.

$$\phi_0(x) = A_q e^{-\frac{\beta^2 x^2}{2}}$$

Now to get the hamiltonian matrix element we do.

$$h_{00} = \int_{-a}^{a} \phi_0^*(x)\hat{H}\phi_0(x)dx$$

Putting $a = 1, m = 1$ and $\hbar = 1$ to work in the Energy units of $\frac{\hbar^2}{ma^2}$ we get $k = 4 \quad \beta = 4^{1/4}$. Evaluating the integrals at these values we get. $h_{00} = 0.9544$ We can construct the matrix similarly for every value of p and q for the dimension given. Getting eigenvalues from the constructed matrix gives the Energy level in the units of $\frac{\hbar^2}{ma^2}$

For N = 5

$$\begin{bmatrix} 0.954500 & 0.000000 & -0.763548 & 0.000000 & -0.396751 \\ 0.000000 & 2.215608 & 0.000000 & -2.468673 & 0.000000 \\ -0.152710 & 0.000000 & 2.072950 & 0.000000 & -3.085998 \\ 0.000000 & -1.058003 & 0.000000 & 2.146257 & 0.000000 \\ -0.044083 & 0.000000 & -1.714443 & 0.000000 & 3.164405 \end{bmatrix}$$

The eigen values for this matrix is:

$$[0.109385071685, \quad 0.564434747324, \quad 1.09686038844, \quad 3.79742982845, \quad 4.98561012119]$$

Which means the first two energy level are

$$E_0 = 0.109\frac{\hbar^2}{ma^2} \quad E_1 = 0.564\frac{\hbar^2}{ma^2}$$

For N = 10

$$E_0 = 7.24 \times 10^{-7}\frac{\hbar^2}{ma^2} \quad E_1 = 1.102 \times 10^{-5}\frac{\hbar^2}{ma^2}$$

The complete program is □

```
#!/usr/bin/env python3

import numpy as np
import numpy.linalg as LA

from scipy.integrate import quad

import sympy as smp
import sympy.functions as smf

class EnergyLevels():

    #define constants
    ca = 1 # potetial well half width = 1
    cm = 1 # mass
    chc = 1 #hbar = 1
    cv0 = 2*chc**2/(cm*ca**2) # constanv V0 inferred from conditions
    ck = 2*cv0/(ca**2) # harmonic oscillator constant
    cb = (cm*ck/(chc**2))**(1/4.) # beta parameter.

    dimlist = [5,10,20] # dimension list
    #define vars and consts
    b,fi,k,m,p,q,x = smp.symbols('beta,phi,k,m,p,q,x',real=True)

    #special variables
```

```
    hcut = smp.symbols('hbar',real=True)
    ndim = 5
    lim = 1

    # constants substution dictionary. This should only affect the
    # scale of the output value.
    subd = {b:cb,hcut:chc,k:ck,m:cm}

    def __init__(self):
        pass

    def V(self,x): #potential function
        k = self.k
        return smp.Rational(1,2)*k*x**2

    def fai(self,n):
        x = self.x; b = self.b
        of = smp.Rational(1,4)
        oh = smp.Rational(1,2)
        A = (b**2/smp.pi)**of*1/(smp.sqrt(2**n*smf.factorial(n)))
        return (smp.exp(-oh*b**2*x**2)*smf.hermite(n,b*x)) * A

    def H(self,psi):
        hf = smp.Rational(1,2)
        x = self.x; m = self.m; hcut = self.hcut
        hctm = -hf*(hcut**2)/(m)
        return hctm * smp.diff(psi,x,2) + self.V(x)*psi

    def getf(self,p,q):
        php = self.fai(p)
        phq = self.fai(q)
        fx = php * self.H(phq)
        return fx


    def geth(self,p,q):
        fx = self.getf(p,q)
        fx = fx.subs(self.subd)

        hpq = self.integrate(fx,-self.ca,self.ca)
        return  hpq

    def integrate(self,fx,a,b):
        flx = smp.utilities.lambdify(self.x,fx)
        v,e = quad(flx,-self.lim,self.lim)
        return v

    def construct_H(self,n):
        H = np.zeros(n*n).reshape(n,n)

        for v in [(p,q) for p in range(n) for q in range(n)]:
            p,q = v
            H[p][q] = self.geth(p,q)

        return np.matrix(H)

    def get_energy(self):
        for m in self.dimlist:
            hmn = self.construct_H(m)
            evl,evc = LA.eig(hmn)
            evl = np.squeeze(evl)
            evl = sorted(evl)
            print(evl)
            #print('For N = {} E0 = {:.3e} and E1 = {:.2e}'.format(m,evl[0],evl[1]))


if __name__ == '__main__':
    EL = EnergyLevels()
```

```
EL.get_energy()
```

1.3 Homework Four

1.3.1. Use contour integration to compute the integral

$$I = \int_{-1}^{1} \frac{dx}{(a^2+x^2)\sqrt{1-x^2}}$$

where a is real and the integrand has a branch cut running from -1 to 1. Sketch the contour you have chosen and carefully justify your reasoning to evaluate or neglect each portion of the total integral.
Solution:
We can write the above integral as

$$\oint \frac{dz}{(a^2+z^2)\sqrt{1-z^2}}$$

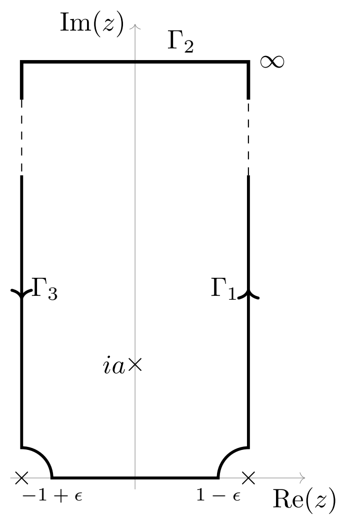


Figure 1.1: There are poles at ± 1 and $\pm ia$ .Since the function is even the integral along two vertical lines will be equal and opposite and vanish. The integral along the bottom horizontal line is what we want, and the integral along the top horizontal line will vanish because at large value of z; $\frac{1}{(a^2+z^2)\sqrt{1-z^2}} = 0$. In the closed contour integral only leaves the integral along the x axis from -1 to 1.

$$\oint f(z)dz = I - 2\pi i \frac{1}{4}\text{Res}(1) + \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \int_{\Gamma_3} f(z)dz - 2\pi i \frac{1}{4}\text{Res}(-1) \tag{1.2}$$

$$\text{Res}f(-1) = \lim_{z\to -1} -\frac{1-z}{(a^2+z^2)\sqrt{1-z^2}} = \lim_{z\to -1} -\frac{\sqrt{1-z}}{(a^2+z^2)\sqrt{1+z}} = 0$$

$$\text{Res}f(1) = \lim_{z\to 1} -\frac{1+z}{(a^2+z^2)\sqrt{1-z^2}} = \lim_{z\to 1} \frac{\sqrt{1+z}}{(a^2+z^2)\sqrt{1+z}} = 0$$

The only terms left in the RHS of Eq. (1.2) is I

$$\begin{aligned} I = \oint \frac{dz}{(a^2+z^2)\sqrt{1-z^2}} &= 2\pi i \text{Res}f(ia) \\ &= 2\pi i \lim_{z\to ia} \frac{\cancel{z-ia}}{(z+ia)(\cancel{z-ia})\sqrt{1-z^2}} \\ &= \frac{2\pi i}{2ia\sqrt{1+a^2}} \\ &= \frac{\pi}{a\sqrt{1+a^2}} \end{aligned}$$

So the required integral is $\int\limits_{-1}^{1} \frac{dx}{(a^2+x^2)\sqrt{1-x^2}} = \frac{\pi}{a\sqrt{1+a^2}}$. □

1.3.2. Work out the details of the contour integral in the context of quantum scattering problem. The problem involves evaluating the integral

$$I(\sigma) = \int\limits_{-\infty}^{\infty} \frac{x \sin x dx}{x^2 - \sigma^2}$$

The integrand has poles on the real axis, and so is only defined as a Cauchy Principal value, deforming the path of integration to avoid the poles using small semicircles of radius ϵ centered on $x = \pm\sigma$. State clearly the assumptions you make and the contours you choose, and show that

$$I(\sigma) = \pi \cos \sigma.$$

Solution:
There are two singular points at $\pm\sigma$. If we write the function as

$$f(z) = \frac{ze^{iz}}{z^2 - \sigma^2}; I(\sigma) = Im \left[\oint f(z)dz \right]$$

Taking this contour, the, integral along the big semicircular contour will go to zero by Jordan's Lemma. The integral along the line includes two semicircular hops.

$$\begin{aligned}
\int\limits_{-R}^{R} f(z)dz &= \overset{-\xi}{\int\limits_{-R}^{-\sigma-\epsilon} f(z)dz} + \overset{2\pi i \frac{1}{2}\text{Res}(-\sigma)}{\int\limits_{-\sigma-\epsilon}^{-\sigma+\epsilon} f(z)dz} + \overset{0,}{\int\limits_{-\sigma+\epsilon}^{\sigma-\epsilon} f(z)dz} + \overset{2\pi i \frac{1}{2}\text{Res}(\sigma)}{\int\limits_{\sigma-\epsilon}^{\sigma+\epsilon} f(z)dz} + \overset{\xi}{\int\limits_{\sigma+\epsilon}^{\sigma-\epsilon} f(z)dz} \\
&= \frac{2\pi i}{2} \left(\text{Res}(-\sigma) + \text{Res}(\sigma)\right) \\
&= \frac{2\pi i}{2} \left[\lim_{z\to-\sigma} \left(\frac{ze^{iz}}{z-\sigma} \right) + \lim_{z\to-\sigma} \left(\frac{ze^{iz}}{z+\sigma} \right) \right] \\
&= \pi i \left[\left(\frac{-\sigma e^{-i\sigma}}{-2\sigma} \right) + \left(\frac{\sigma e^{i\sigma}}{2\sigma} \right) \right] \\
&= \pi i \left[\left(\frac{e^{-i\sigma}}{2} \right) + \left(\frac{e^{i\sigma}}{2} \right) \right] \\
&= \pi i \cos(\sigma)
\end{aligned}$$

Our original integral was $I(\sigma) = Im \left[\oint f(z)dz \right]$ =Im$[\pi i \cos(\sigma)] = \pi \cos(\sigma)$. □

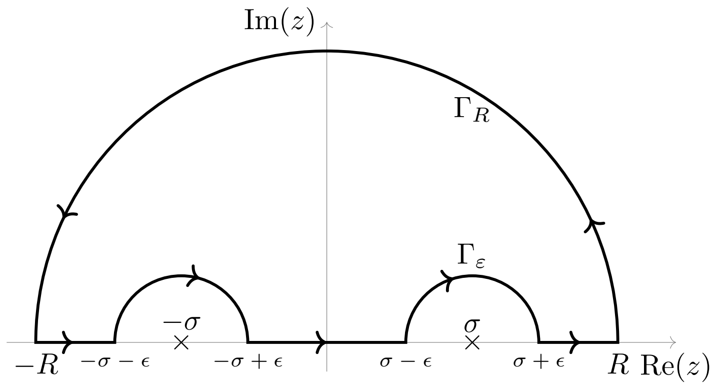
1.3.3. (a) Find the series solution of the equation

$$(1 - x^2)y''(x) - xy'(x) + n^2 y(x) = 0$$

that is regular at $x = 0$. Under what circumstances (for what values of n) does the series converge for *all* x?

Solution:
Let the solution be $y(x) = \sum\limits_{r=0}^{\infty} a_r x^{r+k}$; where $a_0 \neq 0$. Then the first two derivatives are.

$$y'(x) = \sum_{r=0}^{\infty} (r+k) a_r x^{r+k-1}; \qquad y''(x) = \sum_{r=0}^{\infty} (r+k)(r+k-1) a_r x^{r+k-2}$$


Im(z)
Γ_R
Γ_ε
$-\sigma$
σ
$-R$
$-\sigma-\epsilon$
$-\sigma+\epsilon$
$\sigma-\epsilon$
$\sigma+\epsilon$
R Re(z)

Substuting these back into the given differential equation we get.

$$\sum_{r=0}^{\infty}(r+k)(r+k-1)a_r x^{r+k-2} - \sum_{r=0}^{\infty}(r+k)(r+k-1)a_r x^{r+k} - \sum_{r=0}^{\infty}(r+k)a_r x^{r+k} + \sum_{r=0}^{\infty} n^2 a_r x^{r+k} = 0$$

If we take out two terms from the summation sign in the first expression, we get

$$k(k-1)a_0x^{k-2} + k(k+1)a_1x^{k-1} + \sum_{r=2}^{\infty}(r+k)(r+k-1)a_r x^{r+k-2} - \sum_{r=0}^{\infty}(r+k)(r+k-1)a_r x^{r+k} - \sum_{r=0}^{\infty}(r+k)a_r x^{r+k} + \sum_{r=0}^{\infty} n^2 a_r x^{r+k} = 0$$

Since r is a dummy index $\sum\limits_{r=2}^{\infty}(r+k)(r+k-1)a_r x^{r+k-2}$ can be written as $\sum\limits_{r=0}^{\infty}(r+k+2)(r+k+1)a_{r+2}x^{r+k}$

$$k(k-1)a_0x^{k-2} + k(k+1)a_1x^{k-1} + \sum_{r=0}^{\infty}\Big[(r+k+2)(r+k+1)a_{r+2} - (r+k)(r+k-1)a_r - (r+k)a_r x^{r+k} + n^2 a_r\Big] x^{r+k} = 0$$

Since we are expecting solution that is to be true for everyvalue of x every coefficient of each x^{r+k} should go to zero. If it didn't then we would have a polynomial of degree $r+k$ which would give $r+k$ solutions for x and would not be true for any general x other then the solution to it.
Equating the coefficient of x^{k-2} to zero we get $k(k-1)a_0 = 0; \Rightarrow k = \{0,1\}$.
If we choose $k=0$ then the coefficient of x^{k-1} which is $k(k+1)a_1$ goes to zero. So a_1 can be any arbitrary number. If we choose $k=1$ then the coefficient of x^{k-1} which is $k(k+1)a_1 = 0$ requires that $a_1 = 0$. So

$$a_1 = \begin{cases} \text{arbitrary} & \text{if} \quad k=0 \\ 0 & \text{if} \quad k=1 \end{cases}$$

Also the coefficient of x^{r+k} should be zero for every value of $r \geq 0$. Equating the coefficient of $x^{r+k} = 0$ we get

$$(r+k+2)(r+k+1)a_{r+2} = \left((r+k)^2 - n^2\right)a_r; \qquad \Rightarrow a_{r+2} = \frac{(r+k)^2 - n^2}{(r+k+2)(r+k+1)}a_r$$

for k = 0

$$\begin{aligned} a_{r+2} &= \frac{r^2-n^2}{(r+2)(r+1)}a_r \\ a_2 &= \frac{-n^2}{2!}a_0 \\ a_3 &= \frac{1-n^2}{3!}a_1 \\ a_4 &= \frac{2^2-n^2}{4\cdot 3}a_2 = \frac{n^2(n^2-2^2)}{4!}a_0 \\ a_5 &= \frac{3^2-n^2}{5\cdot 4}a_1 = \frac{(n^2-1)(n^2-3^2)}{5!}a_1 \\ &\vdots \qquad \vdots \end{aligned}$$

The solution then is

$$\begin{aligned} y_0(x) &= a_0\left\{1 - \frac{n^2}{2!}x^2 + \frac{n^2(n^2-2^2)}{4!}x^4 + \cdots\right\} \\ &+ a_1\left\{x - \frac{n^2-1}{3!}x^3 + \frac{(n^2-3^2)(n^2-1)}{5!}x^5 + \cdots\right\} \end{aligned}$$

for k=1

$$\begin{aligned} a_{r+2} &= \frac{(r+1)^2-n^2}{(r+3)(r+2)}a_r; \qquad a_1 = 0 \\ a_2 &= \frac{1-n^2}{3!}a_0 \\ a_3 &= \frac{2^2-n^2}{3!}a_1 = 0 \\ a_4 &= \frac{3^2-n^2}{5\cdot 4}a_2 = \frac{(n^2-1)(n^2-3^2)}{5!}a_0 \\ a_5 &= 0 \\ &\vdots \qquad \vdots \end{aligned}$$

The solution then is

$$y_1(x) = a_0\left\{x - \frac{n^2-1}{3!}x^3 + \frac{(n^2-3^2)(n^2-1)}{5!}x^5 + \cdots\right\}$$

The two solution obtained above are linearly dependent so, we will analyze convergence for the first solution. $y_0(x)$ has a form of

$$y_0(x) = a_0\{\text{Even Function of } x\} + a_1\{\text{Odd Function of } x\}$$

Foe n = Even Integer, the Even function will be a n^{th} degree polynoial and similarly for n being odd.
For the convergence of series, we get from the recurrence relation,

$$\lim_{r\to\infty} \frac{t_{r+2}}{t_r} = \lim_{r\to\infty} \frac{a_{r+2}}{a_r} x^2 = \lim_{r\to\infty} \frac{r^2 - n^2}{(r+2)(r+1)} x^2 = x^2$$

For convergence $\lim_{r\to\infty} \frac{t_{r+2}}{t_r} < 1$ which implies that $x^2 < 1; \Rightarrow |x| < 1$ For this series to converge for all values of x, the above ratio should be less than 1 for some value of n, but it doesn't happen for any n. So the series can't be convergent for all values of x. □

(b) Find the series solution of the equation

$$4x^2 y'' + (1 - p^2)y = 0$$

Solution:
Let the solution be $\sum_{r=0}^{\infty} a_r x^{r+k}$ where $a \neq 0$. The Second derivative is

$$y'(x) = \sum_{r=0}^{\infty} (r+k) a_r x^{r+k-1}; \qquad y''(x) = \sum_{r=0}^{\infty} (r+k)(r+k-1) a_r x^{r+k-2}$$

Substuting these back into the given differential equation we get.

$$\sum_{r=0}^{\infty} 4(r+k)(r+k-1) a_r x^{r+k} + \sum_{r=0}^{\infty} (1-p^2) a_r x^{r+k} = 0; \quad \Rightarrow \sum_{r=0}^{\infty} \left[4(r+k)(r+k-1)a_r + (1-p^2)a_r\right] x^{r+k} = 0$$

Since we seek the solution of differential equation which is true for every value of x, it requires that every coefficient of x^{r+k} vanish.

$$\left\{4(r+k)(r+k-1) + (1-p^2)\right\} a_r = 0; \qquad \text{for } r \geq 0$$

Since we suppose $a_0 \neq 0$,

$$4(k+r)(k+r-1) + (1-p^2) = 0; \Rightarrow 4(k+r)^2 - 4(k+r) + (1-p^2) = 0;$$

The solution to the quadratic equation in k has the solution

$$k + r = \frac{4 \pm \sqrt{4^2 - 4 \cdot 4(1-p^2)}}{2 \cdot 4}; \qquad \Rightarrow k + r = \frac{1}{2}(1 \pm p)$$

Putting back the value of $x + r$ in our original solution we get,

$$y(x) = \sum_{r=0}^{\infty} a_r x^{r+k} = \sum_{r=0}^{\infty} a_r x^{\frac{1}{2}[1\pm p]} = \left(\sum_{r=0}^{\infty} a_r\right) x^{\frac{1}{2}[1\pm p]} = \xi x^{\frac{1}{2}[1\pm p]}; \quad \text{Where } \xi = \sum_{r=0}^{\infty} a_r (\text{Constant})$$

So the two independent solution for the equation are $y(x) = \xi_1 x^{\frac{1}{2}(1+p)}$ and $y(x) = \xi_2 x^{\frac{1}{2}(1-p)}$. □

(c) Given the one solution of the differential equation

$$y'' - 2xy' = 0$$

is $y(x) = 1$, use the Wronskian development to find a second, linearly independent solution. Describe the behavior near $x = 0$

Solution:

Comparing with $y'' + p(x)y' + q(x)y = 0$, $p(x) = -2x$ So,

$$\int p(x)dx = -x^2$$

We have $y_1(x) = 1$. The second solution is

$$\begin{aligned} y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1(x)^2} &= \int e^{x^2} dx \\ &= \int \left[1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots\right] dx \\ &= x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \cdots \end{aligned}$$

The function is well defined near $x = 0$. □

1.3.4. A function $f(x)$ is periodic with period 2π, and can be written as a polynomial $P(x)$ for $\pi < x < a$ and as a polynomial $Q(x)$ for $a < x < \pi$. Show that the Fourier coefficients c_n of f go to zero at least as fast as $1/n^2$ as $n \to \infty$if $P(a) = Q(a)$ and $P(\pi) = Q(\pi)$ (i.e. f is continuous), but only as $1/n$ otherwise.

Solution:

The fourier coefficeent is given by.

$$c_n = \int f(x)e^{-inx}dx = \int_{-\pi}^{a} P(x)e^{-inx}dx + \int_{a}^{\pi} Q(x)e^{-inx}dx$$

Integrating by parts

$$\begin{aligned} &= \left[P(x)\frac{e^{-inx}}{-in}\right]_{-\pi}^{a} + \left[Q(x)\frac{e^{-inx}}{-in}\right]_{a}^{\pi} + \int P'(x)\frac{e^{-inx}}{in} + \int Q'(x)\frac{e^{-inx}}{in} \\ &= \frac{1}{n}\left[(P(a) - Q(a))e^{-ina} + (Q(\pi)e^{-in\pi} - P(-\pi)e^{in\pi})\right] + \int P'(x)\frac{e^{-inx}}{in} + \int Q'(x)\frac{e^{-inx}}{in} \\ &\qquad (Q(\pi) - P(-\pi))\cos(n\pi)) \\ &= \frac{1}{n}\left[(P(a) - Q(a))e^{-ina} + (Q(\pi) - P(-\pi))\cos(n\pi))\right] + \int P'(x)\frac{e^{-inx}}{in} + \int Q'(x)\frac{e^{-inx}}{in} \end{aligned}$$

If we continue on this way.

$$\begin{aligned} c_n =& \frac{1}{n}\left[(P(a) - Q(a))e^{-ia} + (Q(\pi) - P(-\pi))\cos(n\pi))\right] + \\ & \frac{1}{n^2}\left[(P'(a) - Q'(a))e^{-ia} + (Q'(\pi) - P'(-\pi))\cos(n\pi))\right] + \cdots + \frac{1}{n^r}\int P^{(r)}(x)\frac{e^{-inx}}{in} + \int Q^{(r)}(x)\frac{e^{-inx}}{in} \end{aligned} \tag{1.3}$$

Let the order of polynomials $P(x)$ and $Q(x)$ be k_1 and k_2 respectively, are polynomials the derivatives will terminate when $r > \max\{k_1, k_2\}$ We will then have a expression for c_n which is a polynomial of $\frac{1}{n}$

If $P(a) = Q(a)$ and $P(-\pi) = Q(\pi)$ the first term of the Eq. (1.3) will vanish and c_n goes at least as $\frac{1}{n^2}$. It can go faster to zero if also the derivatives are equal then second term goes away. If the function do not agree at the boundaries then the first term of the c_n does not vanish and c_n goes only as fast as $\frac{1}{n}$. □

1.3.5. (a) Find the Fourier series $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$ for $-1 < x < 1$ for the sawtooth function

$$f(x) = \begin{cases} -1 - x & (-1 < x < 0) \\ 1 - x & (\ \ 0 < x < 1) \end{cases} \tag{1.4}$$

Solution:
The period of the function is $T = 2$, The fourier coefficient can be calculated as

$$\begin{aligned} b_n = \frac{2}{T}\int f(x) sin(n\pi x) &= -\int_{-1}^{0} (1+x)\sin(n\pi x)dx + \int_{0}^{1} (1-x)\sin(n\pi x)dx \\ &= -\left[-\frac{1}{n\pi} + \frac{\cos(n\pi)}{n\pi}\right] + \left[\frac{1}{n\pi} + \frac{\cos(n\pi)}{n\pi}\right] \\ &= \frac{2}{n\pi} \end{aligned}$$

So the series solution is $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x)$. □

(b) Plot the partial sums $S_N(x) = \sum_{n=1}^{N} b_n \sin(n\pi x)$ of the series for $0 \le x \le 1$, in steps of $\delta x = 0.0005$, and $N = 1, 5, 10, 20, 50, 100$ and 500. What is the maximum overshoot of Fourier series in the case $N = 500$, and at what value of x does it occur?
Solution:
The maximum overshoot for $N = 500$ occurs at $x = 0.0020$ and the value of overshoot is 0.1790. □

1.4 Homework Five

1.4.1. Use contour integration to find the inverse Fourier transform $f(t)$ of the function

$$F(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega}$$

(where $a > 0$), for all values of t. Recall that F was obtained as a Fourier transform of a step function with a discontinuity at $|t| = a$. What is the value of $f(a)$? (Determine $f(a)$ from the integral – don't appeal to the integral properties of Fourier Transforms!).
Solution:
Writing it as

$$\begin{aligned} f(t) &= \mathcal{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega}e^{-i\omega t}d\omega = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{e^{i\omega a} - e^{-i\omega a}}{2i\omega}e^{-i\omega t}d\omega \\ &= \frac{1}{2i\pi}\left[\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(a-t)\omega}}{\omega}d\omega}_{I_1} - \underbrace{\int_{-\infty}^{\infty}\frac{e^{-i(a+t)\omega}}{\omega}d\omega}_{I_2}\right] \\ &= \frac{1}{2i\pi}\left[I_1 - I_2\right] \end{aligned} \tag{1.5}$$

Considering the integral

$$A = \oint_C \frac{e^{i(a-t)z}}{z}dz = \int_{\Gamma_R}\frac{e^{i(a-t)z}}{z}dz + \int_{\Gamma_\epsilon}\frac{e^{i(a-t)z}}{z}dz + \underbrace{\int_{-R}^{-\epsilon}\frac{e^{i(a-t)\omega}}{\omega}d\omega + \int_{\epsilon}^{R}\frac{e^{i(a-t)}}{\omega}d\omega}_{I_1}$$

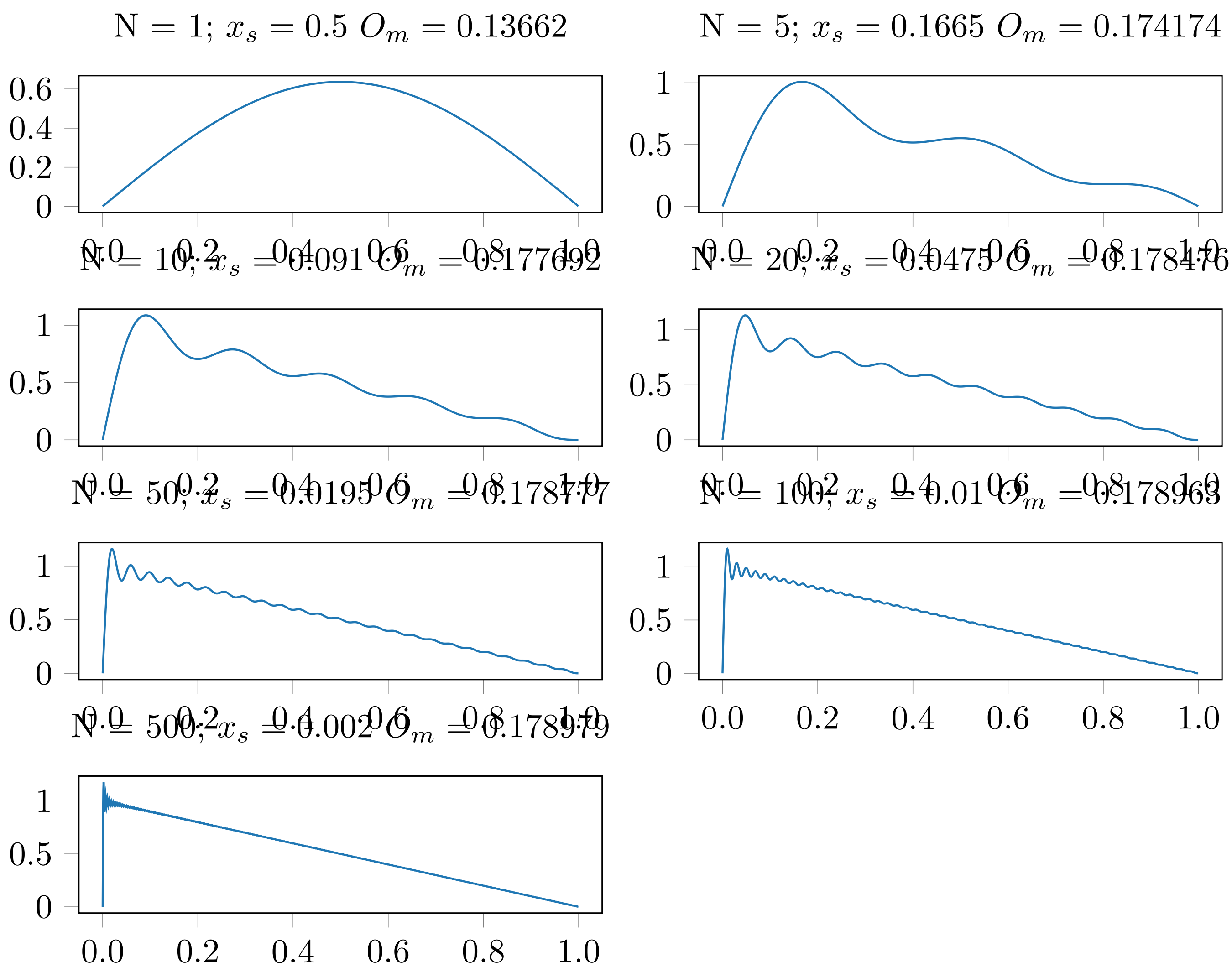
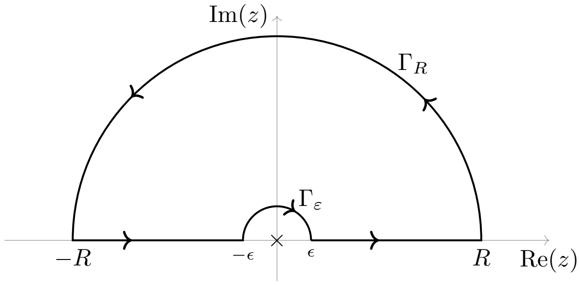


Figure 1.2: Partial Fourier series plot for Eq.(1.4) ($\sum_{n=1}^{N} b_n \sin(n\pi x)$) for different N with Max overshoot of O_m at x_s

If we take limit as $R \to \infty$ and $\epsilon \to 0$ the last two terms of the integral converge to the integral along the ω axis. If the contour is in the upper half of the plane, then the first term of above integral goes to zero by Jordans Lemma if $(a - t) > 0$. But if $(a - t) < 0$ then the integral goes to zero only if the contour is in the lower half of the z plane

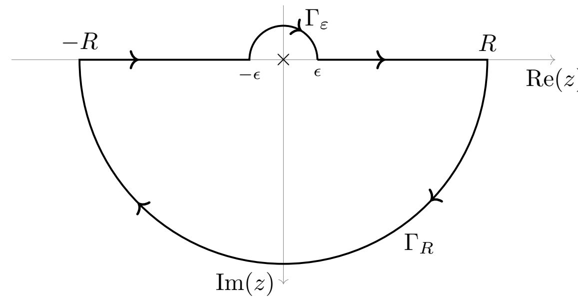
If $\mathbf{a - t > 0}$; $\mathbf{t < a}$



As seen above

$$A = I_1 + \underbrace{\int_{\Gamma_R} \frac{e^{i(a-t)z}}{z} dz}_{0 \text{ By Jordan's Lemma}} + \int_{\Gamma_\epsilon} \frac{e^{i(a-t)z}}{z} dz = 0$$
$$\Rightarrow I_1 - \frac{1}{2} 2\pi i \text{Res} f(0) = 0$$
$$\Rightarrow I_1 - \pi i \lim_{z\to 0} z \frac{e^{i(a-t)z}}{z} = 0$$
$$\Rightarrow I_1 = \pi i \tag{1.6}$$

If $\mathbf{a - t < 0}$; $\mathbf{t > a}$



Also we can see

$$A = I_1 + \underbrace{\int_{\Gamma_R} \frac{e^{i(a-t)z}}{z} dz}_{0 \text{ By Jordan's Lemma}} + \int_{\Gamma_\epsilon} \frac{e^{i(a-t)z}}{z} dz = -2\pi i \text{Res} f(0)$$
$$\Rightarrow I_1 - \frac{1}{2} 2\pi i \text{Res} f(0) = -2\pi i \text{Res} f(0)$$
$$\Rightarrow I_1 = -\pi i \text{Res} f(0)$$
$$\Rightarrow I_1 = -\pi i \lim_{z\to 0} z \frac{e^{i(a-t)z}}{z} = -\pi i \tag{1.7}$$

If $t = a$ (with contour on top half,) then

$$A = I_1 + \int_{\Gamma_R} \frac{e^{i(a-t)z}}{z} dz + \int_{\Gamma_\epsilon} \frac{e^{i(a-t)z}}{z} dz = I_1 + \underbrace{\int_{\Gamma_R} \frac{1}{z} dz}_{\pi i \text{Res} f(0)} + \underbrace{\int_{\Gamma_\epsilon} \frac{1}{z} dz}_{-\pi i \text{Res} f(0)} = 0; \Rightarrow I_1 = 0 \tag{1.8}$$

From (1.6) and Eq. (1.7) and Eq. (1.8) we get

$$I_1 = \begin{cases} \pi i & \text{if } a - t > 0 \\ 0 & \text{if } t = a \\ -\pi i & \text{if } a - t < 0 \end{cases} \tag{1.9}$$

Considering the integral

$$B = \oint_C \frac{e^{-i(a+t)z}}{z} dz = \int_{\Gamma_R} \frac{e^{-i(a+t)z}}{z} dz + \int_{\Gamma_\epsilon} \frac{e^{-i(a+t)z}}{z} dz + \underbrace{\int_{-R}^{-\epsilon} \frac{e^{-i(a+t)\omega}}{\omega} d\omega + \int_{\epsilon}^{R} \frac{e^{-i(a+t)}}{\omega} d\omega}_{I_2} = 0$$

By similar arguments

$$I_2 = \begin{cases} \pi i & \text{if } a + t < 0 \\ 0 & \text{if } a + t = 0 \\ -\pi i & \text{if } a + t > 0 \end{cases} \tag{1.10}$$

From Eq. (1.9) and Eq. (1.10) we get.

$$\begin{aligned}
&\text{if } t < -a; && I_1 = \pi i \text{ and } I_2 = \pi i \Rightarrow I_1 - I_2 = 0; \quad f(t) = \frac{1}{2\pi i}[I1 - I2] = 0 \\
&\text{if } t = -a; && I_1 = \pi i \text{ and } I_2 = 0 \Rightarrow I_1 - I_2 = \pi i; \quad f(t) = \frac{1}{2\pi i}[I1 - I2] = \frac{1}{2} \\
&\text{if } -a < t < a; && I_1 = \pi i \text{ and } I_2 = -\pi i \Rightarrow I_1 - I_2 = 2\pi i; \quad f(t) = \frac{1}{2\pi i}[I1 - I2] = 1 \\
&\text{if } t = a; && I_1 = 0 \text{ and } I_2 = -\pi i \Rightarrow I_1 - I_2 = \pi i; \quad f(t) = \frac{1}{2\pi i}[I1 - I2] = \frac{1}{2} \\
&\text{if } t > a; && I_1 = -\pi i \text{ and } I_2 = -\pi i \Rightarrow I_1 - I_2 = 0; \quad f(t) = \frac{1}{2\pi i}[I1 - I2] = 0
\end{aligned}$$

Combining all these we get

$$f(t) = \begin{cases} 1 & |t| < a \\ \frac{1}{2} & |t| = a \\ 0 & |t| > a \end{cases}$$

So the value of $f(a)$ is $\frac{1}{2}$ from the inverse fourier transform. □

1.4.2. Find the $3-D$ Fourier transform of the wave function for a $2p$ electron in a hydrogen atom:

$$\psi(\boldsymbol{x}) = (32\pi a^5)^{-1/2} z e^{-r/2a_0}$$

where $a = \frac{\hbar^2}{me^2}$ is the Bohr radius, r is radius, and z is a rectangular corrdinate.

Solution:

Supposing $A = (32\pi a^5)^{-1/2}$ and in spherical coordinate system $z = r\cos(\theta)$. Also the volume element in spherical system is $d^3r = r^2 \sin(\theta) d\phi d\theta$ Also *due to spherical symmetry we can write* $\boldsymbol{k} \cdot \boldsymbol{r} = kr\cos(\theta)$*[Riley and Hobson pp 906]* The fourier transform is then

$$\Psi(k) = \frac{1}{\sqrt{(2\pi)^3}} \int_0^\infty rcos(\theta) e^{r/2a} e^{-ikr\cos(\theta)} d^3r = \frac{2\pi}{\sqrt{(2\pi)^3}} \int_0^\pi \int_0^\infty r^3 e^{r/2a} \sin(\theta)\cos(\theta) e^{ikr\cos(\theta)} d\theta dr$$

$$\Psi(k) = \frac{A}{\sqrt{(2\pi)^3}} \int_0^\infty dr\, r^3\, e^{-r/(2a)} \int_0^\pi d\theta\, \sin\theta \cos\theta\, e^{ikr\cos\theta} = \frac{A}{\sqrt{(2\pi)^3}} \int_0^\infty dr\, r^3\, e^{-r/(2a)} \int_\pi^0 d(\cos\theta) \cos\theta e^{ikr\cos\theta}$$

Supposing $krcos(\theta) = u$ $du = sin(\theta)kdr$

$$\begin{aligned}
&= \frac{A}{\sqrt{(2\pi)^3}} \int_0^\infty dr\, r^3\, e^{-r/(2a)} \left[-i \frac{\partial}{\partial(kr)} \int_{-1}^1 du\, e^{ikru} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty dr\, r^3\, e^{-r/(2a)} \frac{\partial}{\partial(kr)} \frac{\sin kr}{kr} \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty dr\, r^3\, e^{-r/(2a)} \left[\frac{\cos kr}{kr} - \frac{\sin kr}{(kr)^2} \right]
\end{aligned}$$

The integral of this function can be obtained with contour integral

With substitution $z = 1/(2a) - ik$ and $cos(kr) = \mathrm{Re}e^{ikr}$

$$\begin{aligned}
\Psi(k) &= A\sqrt{\frac{2}{\pi}} \frac{2}{k} \mathrm{Re} \left[\frac{1}{\left(\frac{1}{2a} - ik\right)^3} \right] - A\sqrt{\frac{2}{\pi}} \frac{1}{k^2} \mathrm{Im} \left[\frac{1}{\left(\frac{1}{2a} - ik\right)^2} \right] \\
&= A\sqrt{\frac{2}{\pi}} \frac{2}{k} \frac{\frac{1}{8a^3} - \frac{3k^2}{2a}}{\left(\frac{1}{4a^2} + k^2\right)^3} - A\sqrt{\frac{2}{\pi}} \frac{1}{k^2} \frac{\frac{k}{a}}{\left(\frac{1}{4a^2} + k^2\right)^2} \\
&= -A\sqrt{\frac{2}{\pi}} 256a^4 \frac{ka}{\left(1 + 4k^2a^2\right)^3}
\end{aligned}$$

This gives the fourier transform of the function. □

1.4.3. Consider the solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + xy = 0$$

for which $|y| \to 0$ as $|x| \to \infty$. (This is the *Airy equation*. It appears in the theory of the diffaraction of light.)

(a) Sketch the solution. Don't use Mathematica!. Specifically, what behavior do you expect as $x \to -\infty$ and $x \to +\infty$?
Solution:

□

(b) By fourier transforming the above equation, determine $Y(\omega)$, te Fourier transform of $y(x)$, and hence write down an integral expression for $y(x)$. (Hint: What is the inverse transform of $Y'(\omega)$)
Solution:
Let us suppose that the fourier transform of $y(x)$ is $Y(\omega)$. The fourier transform can be written as

$$\mathcal{F}(y(x)) = Y(\omega) = \int\limits_{-\infty}^{\infty} y(x)e^{-i\omega x}dx$$

.
Taking derivative of both sides with respect to ω

$$Y'(\omega) = \frac{d}{d\omega}\int\limits_{-\infty}^{\infty} y(x)e^{-i\omega x}dx = \int\limits_{-\infty}^{\infty} y(x)\frac{\partial}{\partial\omega}\left(e^{-i\omega x}\right)dx = -i\int\limits_{-\infty}^{\infty} xy(x)e^{-i\omega x}dx = -i\mathcal{F}(xy) \quad (1.11)$$

Taking the fourier transform of both sides of given differential equation we get,

$$y''(x) + xy = 0; \Rightarrow \mathcal{F}\left[\frac{d^2y}{dx^2}\right] + \mathcal{F}(xy) = \mathcal{F}(0);$$

Using the property of fourier transform $\mathcal{F}(y'') = (-i\omega)^2\mathcal{F}(y)$ the fourier transform and using $\mathcal{F}(xy)$ from Eq. (1.11) we get.

$$(-i\omega)^2Y(\omega) - iY'(\omega) = 0; \Rightarrow \frac{Y'(\omega)}{Y(\omega)} = -i\omega^2; \Rightarrow \int\frac{Y'(\omega)}{Y(\omega)}d\omega = \int -i\omega^2 d\omega; \Rightarrow Y(\omega) = e^{-i\frac{\omega^3}{3}}$$

The solution for the Airy equation which is our original differential equation is just the inverse fourier transform of this equation.

$$y(x) = \mathcal{F}^{-1}\left[Y(\omega)\right] = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{\frac{-i\omega^3}{3}} e^{-i\omega x} d\omega$$

This gives the integral expression for the solution of the differential equation required. □

1.4.4. Find the Green's function $G(x, x')$ for the equation

$$\frac{d^2y}{dx^2} - k^2 y = f(x)$$

for $0 \le x \le l$, with $y(0) = y(l) = 0$.

Solution:

The green's function solution to non homogenous differential equation $\mathcal{L}y = f(x)$ is a solution to homogenous part of the differential equation with the source part replaced as delta function $\mathcal{L}y = \delta(x - x')$. The ontained solution is $G(x, x')$, i.e., $\mathcal{L}G(x, x') = \delta(x - x')$. This solution corresponds to the homogenous part only as it is independent of any source term $f(x)$.

$$\frac{d^2}{dx^2}G(x, x') - k^2 y = \delta(x - x'); \qquad \text{with } G(0, x') = 0; \text{ and } G(l, x') = 0 \text{ for all } 0 \le x' \le l \tag{1.12}$$

Since delta function $\delta(x - x')$ is zero everywhere except $x = x'$ we can find solution for two regions $x < x'$ and $x > x'$. For $x < x'$ let the solution to $\mathcal{L}y = 0$ be $y_1(x)$ and for $x > x'$ be $y_2(x)$ then

$$y_1''(x) - k^2 y_1(x) = 0; \text{ for } x < x'; \qquad y_2''(x) - k^2 y_2(x) = 0; \text{ for } x > x'$$

These are well known harmonic oscillator equations whose solution are

$$y_1(x) = Asin(kx) + Bcos(kx); \qquad y_2(x) = Csin(kx) + Dcos(kx)$$

By the boundary condition $y_1(0) = 0$ and $y_2(l) = 0$. These immediately imply that $B = 0$. Also since the soution to the differential equation must be continuous $y_1(x') = y_2(x')$. Also integrating Eq. (4.28) in the vicinity of x' we get

$$y'(x)\Big|_{x'_-}^{x'_+} - k^2 \overbrace{\int y dx\Big|_{x'_-}^{x'_+}}^{0 \text{ By contunity}} = \int_{x'_-}^{x'_+} \delta(x - x')dx; \Rightarrow y'(x'_+) - y'(x'_-) = 1$$

From three different conditions, (i) contunity at x', (ii) $y_2(l) = 0$ and (iii) $y_1'(x') - y_2'(x') = 1$ we get following three linear equations. Using there parameters we get.

$$\begin{aligned} Ck\cos(kx') - Dk\sin(kx') - Ak\cos(kx') &= 1 \\ C\sin(kx') + D\cos(kx') - A\sin(kx') &= 0 \\ C\sin(kl) + D\cos(kl) &= 0 \end{aligned}$$

Which can be written in the matrix form and solved as.

$$\begin{bmatrix} k\cos(kx') & -k\sin(kx') & -k\cos(kx') \\ \sin(kx') & \cos(kx') & -\sin(kx') \\ \sin(kl) & \cos(kl) & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ A \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C \\ D \\ A \end{bmatrix} = \begin{bmatrix} \frac{\sin(kx')}{k\tan(kl)} \\ -\frac{1}{k}\sin(kx') \\ -\frac{\sin(k(l-x'))}{k\sin(kl)} \end{bmatrix}$$

Giving

$$C = \frac{\sin(kx')}{k\tan(kl)}; \qquad D = -\frac{1}{k}\sin(kx'); \qquad A = -\frac{\sin(k(l - x'))}{k\sin(kl)}$$

So the required function is

$$G(x,x') = \begin{cases} y_1(x) = -\frac{\sin\left(k\left(l-x'\right)\right)}{k\sin(kl)}\sin(kx) & \text{if } x < x' \\ y_2(x) = \frac{sin(kx')}{k}\left(\frac{\sin(kx)}{\tan(kl)} - \cos(kx)\right) & \text{if } x > x' \end{cases} \tag{1.13}$$

Eq.(4.29) gives the Green's function whcin can be used to find the solution to the differential equation

$$y(x) = \int G(x,x')f(x')dx'$$

The solution to the original inhomogenous differential equation can is given by the above expression in terms of Green's function. □

1.4.5. Poisson's equation (in three dimensions) is $\nabla^2\phi = 4\pi G\rho$

(a) Let $\widetilde{\phi}(\boldsymbol{k})$ be the fourier transforms of $\phi(x)$ and $\rho(x)$, respectively show that:

$$\widetilde{\phi} = -\frac{4\pi G\widetilde{\rho}}{k^2}$$

and hence write down an integral expression for $\phi(x)$.
Solution:
Taking the fourier transform ov Poissons equation we have

$$4\pi G\mathcal{F}(\rho(r)) = \int_{-\infty}^{\infty} \nabla^2\phi(r)e^{ik\cdot r}d^3r$$

Wriging in cartesian coordinate system $\boldsymbol{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k}$ and $\boldsymbol{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ we have

$$\begin{aligned} 4\pi G\widetilde{\rho}(\boldsymbol{k}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\boldsymbol{r})\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)e^{xk_x+yk_y+zk_z}dxdydz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\boldsymbol{r})\left((ik_x)^2 + (ik_y)^2 + (ik_z)^2\right)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}dxdydz \\ &= (-k_x^2 - k_y^2 - k_z^2)\mathcal{F}(\phi(\boldsymbol{r})) = -\left|\boldsymbol{k}\right|^2\widetilde{\phi}(\boldsymbol{k}); \\ \Rightarrow \widetilde{\phi}(\boldsymbol{k}) &= -\frac{4\pi G\widetilde{\rho}(\boldsymbol{k})}{k^2} \end{aligned}$$

This gives the expression for the fourier transform for Poisson's equation. This can be used to get the expression of $\phi(\boldsymbol{x})$ which is

$$\phi(\boldsymbol{x}) = \frac{4\pi G}{\sqrt{(2\pi)^3}}\int_{-\infty}^{\infty}\frac{1}{k^2}\tilde{\rho}(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}d^3k \tag{1.14}$$

This is the expression for $\phi(x)$ which is the solution to Poisson's equation. □

(b) For a point mass at the origin, $\rho(x) = m\delta(x)$. Use the above to determine the expression for $\phi(x)$
Solution:
Taking the fourier transform of given density function

$$\tilde{\rho}(k) = \int_{-\infty}^{\infty} m\delta(\boldsymbol{r})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d^3r = m; \qquad \text{Integral of delta function is 1}$$

Substuting this in Eq. (1.14) we get

$$\phi(\boldsymbol{x}) = \frac{4\pi G}{\sqrt{(2\pi)^3}} \int_{-\infty}^{\infty} \frac{m}{k^2} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d^3k = \frac{4\pi G m}{\sqrt{(2\pi)^3}} \int_{-\infty}^{\infty} k^{-2} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d^3k$$

This integral should give $\phi(x) = -\frac{Gm}{x}$ for $x > 0$ □

1.5 Homework Six

1.5.1. The function $f(t_k)$ adn $F(\omega_n)$ are discrete Fourier transforms of one anotherr, where $t_k = k\Delta, \omega_n = 3\pi n/N\Delta$, for $k, n = 0, \cdots, N-1$. Show that.

(a) if f is real, then $F(\omega_n) = F^*(4\pi f_c - \omega_n)$,
(b) if f is pure imaginary, then $F(\omega_n) = -F^*(4\pi f_c - \omega_n)$,

where $f_c = 1/2\Delta$ is the Nyquist frequency.
Solution:
The fourier discrete fourier transform fo $f(t_k)$ by definition is

$$F(\omega_n) = \sum_{k=0}^{N-1} f(t_k) e^{i\omega_n t_k} = \sum_{k=0}^{N-1} f(k\Delta) e^{i\frac{2\pi n}{N\Delta}k\Delta} = \sum_{k=0}^{N-1} f(k\Delta) e^{2\pi i n k/N}$$

Taking conjugate of the above expression with ω_n replaced by $4\pi f_c - \omega_n$ we get.

$$F^*(4\pi f_c - \omega_n) = \sum_{k=0}^{N-1} f^*(k\Delta) \left(e^{i\left(4\pi f_c - \frac{2\pi n}{N\Delta}\right)k\Delta}\right)^* = \sum_{k=0}^{N-1} f^*(k\Delta) \left(\overbrace{e^{-2i\pi}}^{1} {}^{+2i\pi n/N}\right)^k = \sum_{k=0}^{N-1} f^*(k\Delta)(e^{2\pi i n k/N})$$

if f is real then $f^*(k\Delta) = f(k\Delta)$

$$F^*(4\pi f_c - \omega_n) = \sum_{k=0}^{N-1} f(k\Delta)(e^{2\pi i n k/N}) = F(\omega_n)$$

if f is pure imaginary then $f^*(k\Delta) = -f(k\Delta)$

$$F^*(4\pi f_c - \omega_n) = \sum_{k=0}^{N-1} -f(k\Delta)(e^{2\pi i n k/N}) = -F(\omega_n)$$

Which completes the proof. □

1.5.2. (a) Let $\mathcal{R}_j$ be a random sequence of real numbers, with $\mathcal{R}_j$ distributed uniformly between -1 and 1. For N-piont discrete Fourier transform of $\mathcal{R}_j$: $r_k = \sum_{j=0}^{N-1} \mathcal{R}_j e^{2\pi i j k/N}$, calculate the expectation value and variance of the "periodogram estimate" of the power spectrum, $P_k = |r_k|^2 + |r_{N-k}|^2$, for $k = 1, \cdots, N/2$.
Solution:
Given $P_k = |r_k|^2 + |r_{N-k}|^2$ we can calculate the expectation value of the function P_k as

$$\langle P_k \rangle = \frac{1}{N}\left[\sum_{1}^{N/2} |r_k|^2 + \sum_{1}^{N/2} |r_{N-k}|^2\right] = \frac{1}{N}\sum_{0}^{N-1} |r_k|^2 \overset{\text{Parseval's Theorem}}{=} \sum_{0}^{N-1} |\mathcal{R}_k|^2$$

This the expectation value of the function P_k Now for the variance

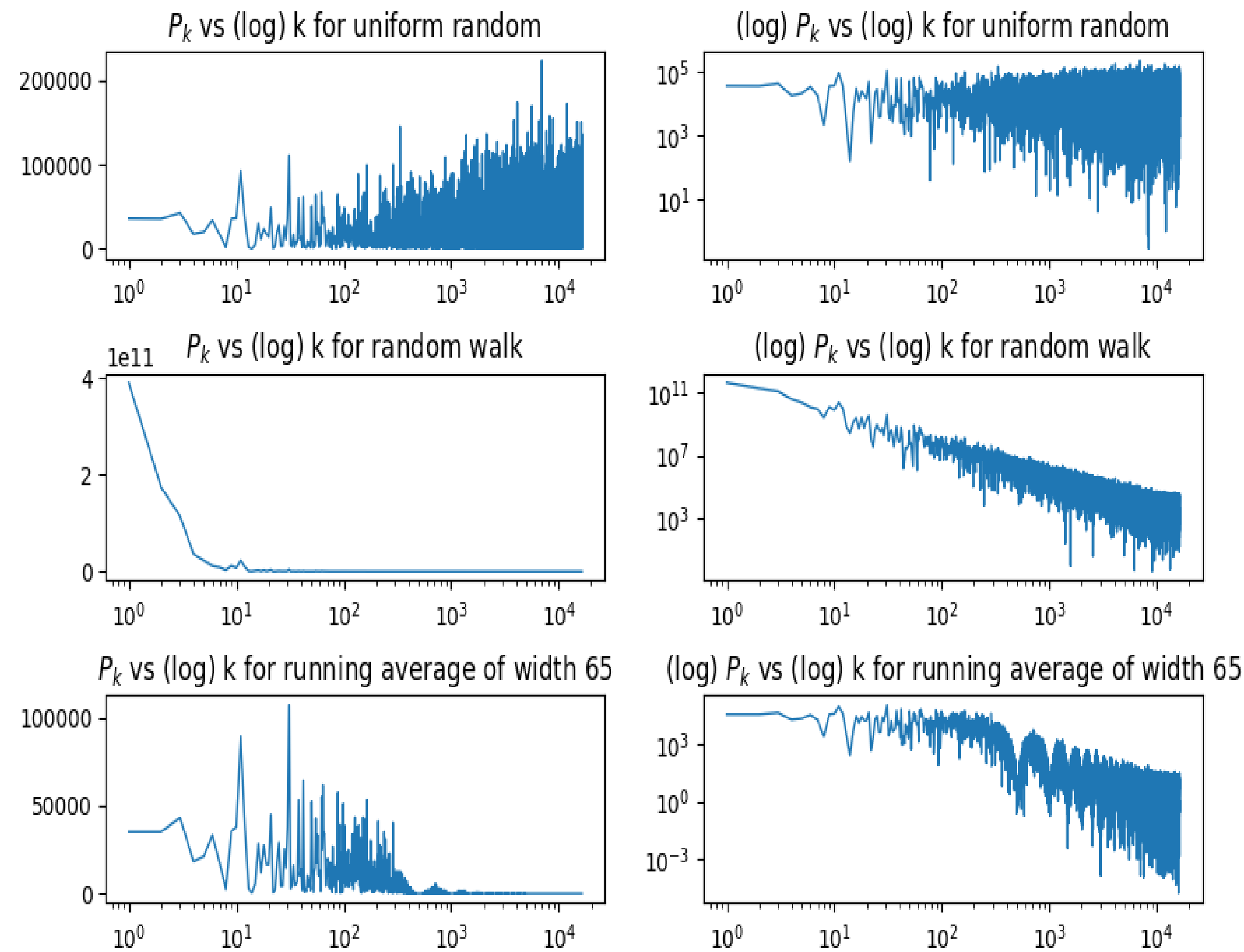
$$Var(P_k) = \langle P_k^2 \rangle - \langle P_k \rangle^2 = \frac{1}{N}\sum_{1}^{N/2} \left[|r_k|^2 + |r_{N-k}|^2\right]^2 - \sum_{0}^{N-1} |\mathcal{R}_k|^2$$

The simplification should give the variance. □

(b) Generate a sequence of random numbers with properties as in part (a), and compute P_k numerically using a fast Fourier transform with $N = 32768$. Plot first P_k, then $\log_{10} P_k$ against $\log_{10} k$, for $k = 1, \cdots, N/2$. How does this graph compater with the analytic expectations from part (a)? Repeat the calcluations, averaging the data over an interval of width 65 centered on each frequency data point, and plot the results.

(c) Repeat the computation in part (b) for *random walk*w_j definde by $w_0 = 0; \quad w_{j+1} = w_j + \mathcal{R}_j$. Can you account for the differences in appearence between this graph and one you obtained in part (b)?

Solution:
For the graph of uniform random the power is flat curve for up to a high frequency range but



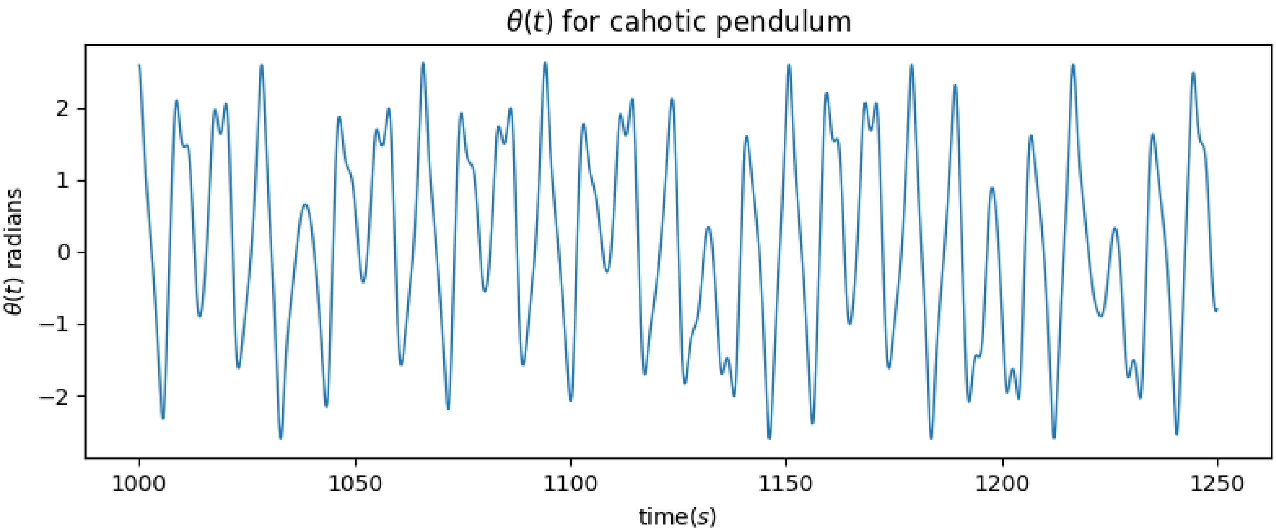
for random walk the power at higher frequency is significantly lower than the power at lower frequencies. □

1.5.3. The file `pendulum2.dat` contains a chaotic data set generated as a solution to the equations of motion of the damped, driven, nonlinear pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{1}{q}\frac{d\theta}{dt} + sin(\theta) = g\cos(\omega_d t).$$

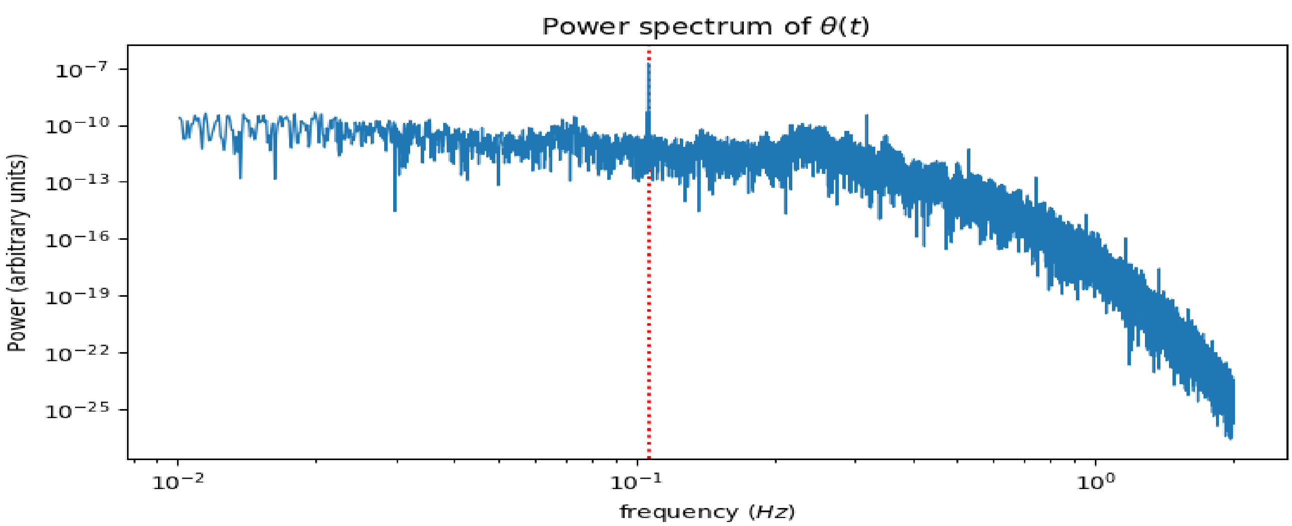
Contains four column t, θ, ω and $\phi = \omega_d t$

(a) Plot the time sequence (t) for 1000 t 1250.
Solution:



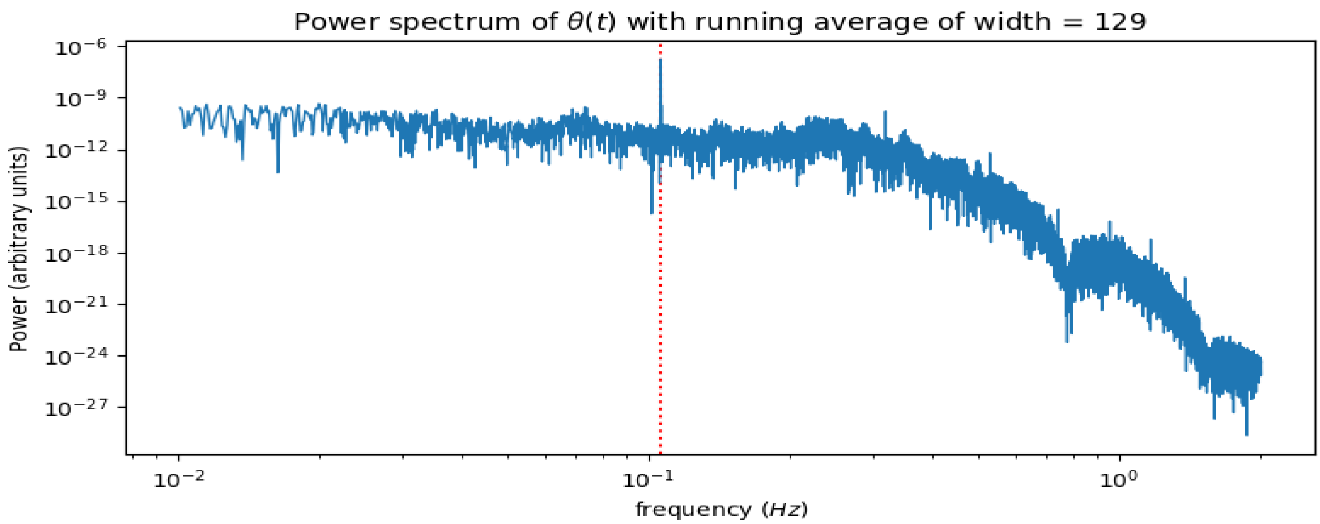
This graph shows the plot of $\theta(t)$ vs t for the time range $t = 1000$ to $t = 1250$ □

(b) Use an FFT to compute the power spectrum $P(f)$ of $\theta(t)$, where f is frequency. Use the entire dataset, with a Bartlett data window, and plot $P(f)$ with log–log axes for $0.01 \leq f \leq 2$. Can you identify any features in the plot?
Solution:

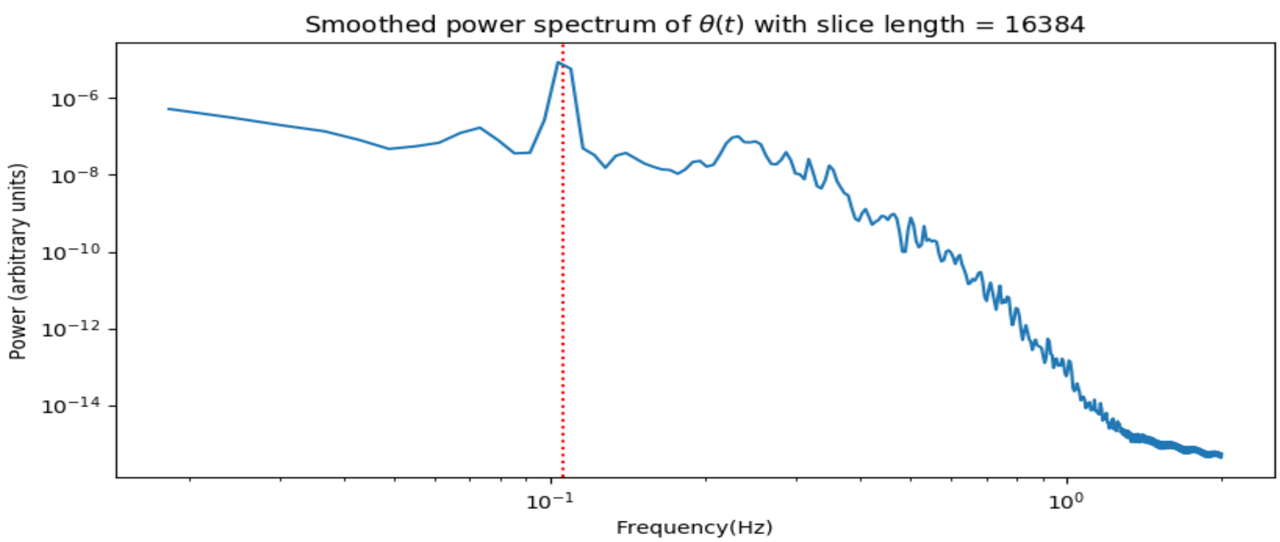


There is a sharp spike at frequency $f = 0.106$ which equals the frequency of the driving force. $w_d = 2/3$

(c) As in Problem 2, smooth the data by averaging over an interval of width 129 centered on each frequency data point, and plot the results as in part (b).



(d) Implement the alternative smoothing strategy of dividing the input dataset into a series of slices each of length 16384, computing the power spectra of each, and then averaging all the individual power spectra. Again plot the results as in part (b). Don't forget the Bartlett windows!

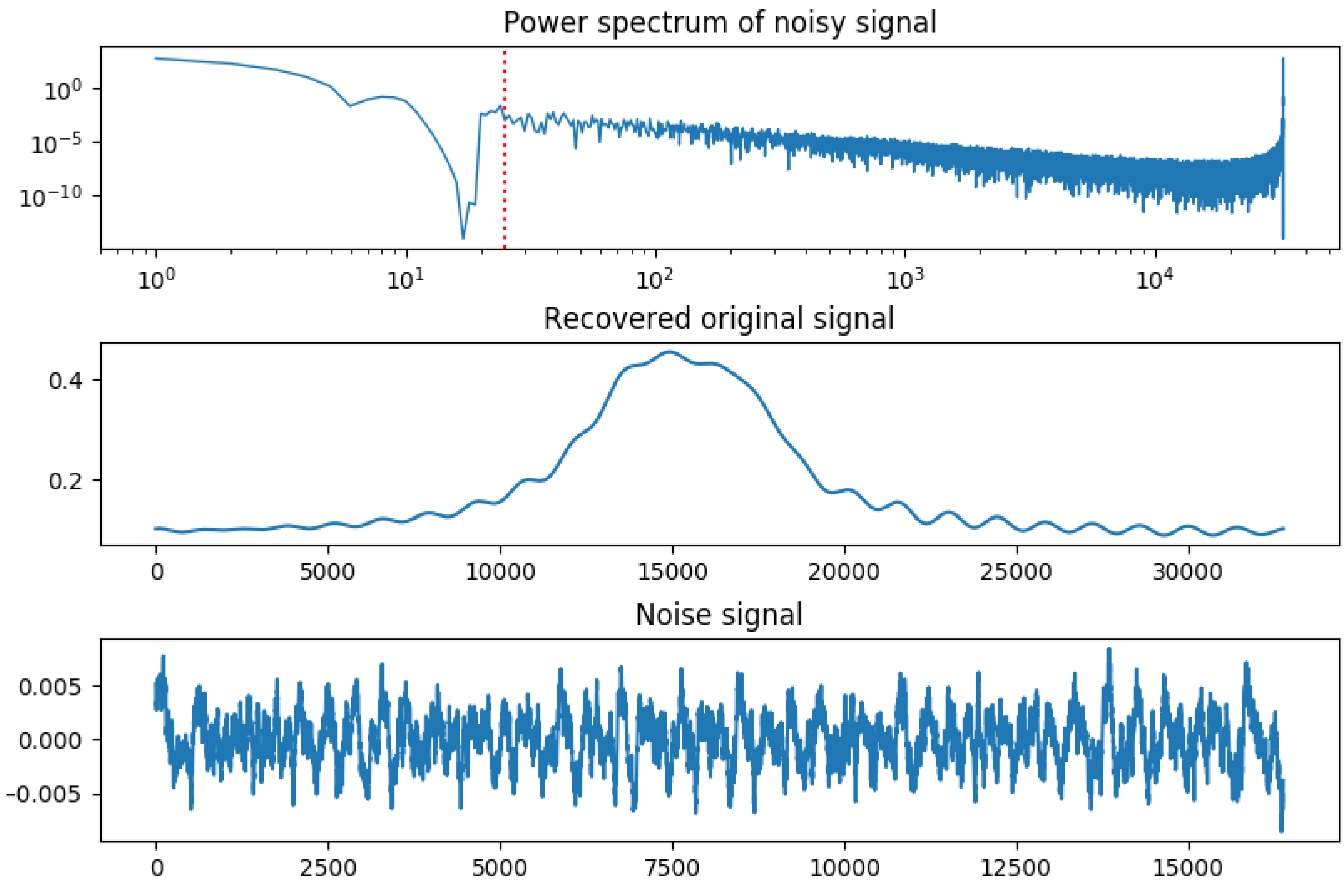


On each of these power spectrum there the spectrum is flat for lower frequency and it drops sharply after some frequency. A common feature in all of these graphs is the presence of a spike in power spectrum at $f = 0.106$. This corresponds to the frequency of the driving force. □

1.5.4. A "corrupted" real-valued dataset may be found in the file `corrupt.dat`. It is a time sequence consisting of two columns of data, j and c j , for $j = 0, \cdots N1$. The original data have been convolved with a Gaussian transfer function of the form $g \approx exp(j^2/a^2)$ (normalized so that $\sum_j g_j = 1$), with a = 2048, and are subject to random noiseof some sort at some level.

Find a filter to apply to the data, and plot your best-guess reconstruction of the original uncorrupted dataset. Can you characterize the type of noise in the data?

Solution:



The power spectrum of the corrupted signal has high value for lowe frequencies and it is significantly small for higher frquency. At around 25^{th} frequency bin the noise is completely dominates. So I chose 25^{th} frequency bin as the cutoff point.

The recovered signal looks like a gaussian function. The recovered noise looks like a white noise. □

QuestionTwo

```python
#!/usr/bin/env python3

import itertools
import numpy as np
from scipy import signal
import matplotlib.pyplot as plt

class PowerSpectrum():
    def __init__(self,N):
        self.N = N
        self.pcnt = 1
        self.spl = 220
        self.wid = 65

    def get_rnd_sequence(self):
        rnd = np.random.uniform(-1,1,self.N)
        return rnd

    def get_rnd_walk(self,sig):
        wlk = list(itertools.accumulate(sig))
        return wlk

    def get_power(self,sig):
        r = np.fft.fft(sig); N = len(sig)
        Pk = [np.abs(r[k])**2 + np.abs(r[N-k])**2 for k in range
            (1,int(N/2))]
        return Pk

    def plot_spectrum(self,sig,title=''):
        Pk = self.get_power(sig)
        k = range(len(Pk))
        plt.subplot(self.spl+self.pcnt); plt.xscale('log');self.
            pcnt+=1
        plt.plot(k,Pk,lw=1); plt.title(r'$P_k$ vs (log) k for '+
            title)
        plt.subplot(self.spl+self.pcnt); plt.xscale('log'); plt.
            yscale('log');self.pcnt+=1
        plt.plot(k,Pk,lw=1); plt.title(r'(log) $P_k$ vs (log) k
            for '+title)

    def get_run_average(self,sig,wid):
        hwid = int((wid-1)/2)
        ravg = [np.average(sig[k-hwid:k+hwid+1]) for k in range(
            hwid,len(sig)-hwid)]
        return ravg

    def machinge_periodogram(self,sig):
        plt.subplot(self.spl+self.pcnt); self.pcnt +=1
        plt.xscale('log')
        prd,Pxx_den = signal.periodogram(rsig)
        plt.plot(prd,Pxx_den,lw=1)
        plt.subplot(self.spl+self.pcnt); self.pcnt +=1
        plt.xscale('log'); plt.yscale('log')
        plt.plot(prd,Pxx_den,lw=1)

    def new_experiment(self):
        rsig = self.get_rnd_sequence()
        wsig = self.get_rnd_walk(rsig)
        ravg = self.get_run_average(rsig,PS.wid)

        plt.xscale('log'); plt.yscale('log')
        rprd,rpxxden = signal.periodogram(rsig)
        aprd,apxxden = signal.periodogram(ravg)
        plt.plot(rprd,rpxxden)
        plt.plot(aprd,apxxden)
        plt.show()


    def experiment(self):
        rsig = PS.get_rnd_sequence()
        wsig = PS.get_rnd_walk(rsig)
        ravg = PS.get_run_average(rsig,PS.wid)

        plt.xscale('log'); plt.yscale('log')
        plt.plot(self.get_power(rsig))
        plt.plot(self.get_power(ravg))
        plt.show()


if __name__ == '__main__':
    PS = PowerSpectrum(32768)
    PS.spl = 320; PS.wid = 65

    rsig = PS.get_rnd_sequence()
    wsig = PS.get_rnd_walk(rsig)
    ravg = PS.get_run_average(rsig,PS.wid)

    PS.plot_spectrum(rsig, 'uniform random ')
    PS.plot_spectrum(wsig, 'random walk')
    PS.plot_spectrum(ravg, 'running average of width {}'.format(PS
        .wid))
    #PS.machinge_periodogram(rsig)
    #PS.new_experiment()

    plt.show()
```

Question Three

```
#!/usr/bin/env python3

import itertools
import numpy as np
import matplotlib.pyplot as plt

from  matplotlib2tikz import save as tikz_save

class ForcedPendulum():
    def __init__(self,flname):
        self.filename = flname
        self.slc = 20
        self.spl = 210; self.pcnt = 1
        self.read_file()

    def read_file(self):
        content = np.genfromtxt(self.filename)
        self.t     = content[:,0];
        self.theta = content[:,1]
        self.omega = content[:,2]
        self.phi   = content[:,3]
        return content

    def bratlett_window(self,sig):
        N = len(sig); hlen = int(N/2)
        sig2 = np.copy(sig)
        sig2[0:hlen] = [k*sig[k] for k in range(hlen)]
        sig2[hlen:N] = [(N-k)*sig[k] for k in range(hlen,N)]
        wss = N*np.sum([1-np.abs(2*j-N)/N for j in range(N)])
        return sig2/(wss)

    def get_run_average(self,sig,wid):
        hwid = int((wid-1)/2)
        ravg = [np.average(sig[k-hwid:k+hwid+1]) for k in range(
            hwid,len(sig)-hwid)]
        return ravg

    def plot_theta(self,min_time=1000,max_time=1250):
        tmin = min_time; tmax = max_time
        minidx = np.searchsorted(self.t,tmin)+1
        maxidx = np.searchsorted(self.t,tmax)+1
        plt.subplot(self.spl+self.pcnt); self.pcnt += 1; plt.
            tight_layout()
        tdata  = self.t[minidx:maxidx]
        thdata = self.theta[minidx:maxidx]
        plt.plot(tdata,thdata,lw=1)
        plt.title(r'$\theta(t)$ for cahotic pendulum');
        plt.xlabel(r'time$(s)$'); plt.ylabel(r'$\theta(t)$ radians
            ')

    def get_power(self,sig):
        sft = np.fft.fft(sig)
        N = len(sig)
        power = 1/N*(np.abs(sft))**2
        dt = (max(self.t) - min(self.t))/len(self.t)
        dw = 2*np.pi/(N*dt)
        omega = np.arange(N)*dw;

        return power, omega

    def plot_power_spectrum(self,sig,minfrq=None,maxfrq=None):
        power,omega = self.get_power(self.bratlett_window(sig));
            freq = omega/(2*np.pi)
        minfloc = np.searchsorted(freq,minfrq)+1 if minfrq != None
             else 0
        maxfloc = np.searchsorted(freq,maxfrq)+1 if maxfrq != None
             else len(omega)
        maxpowloc = power.argmax(); maxpowfreq= freq[maxpowloc]

        plt.subplot(self.spl+self.pcnt); self.pcnt += 1;plt.
            tight_layout()
        plt.yscale('log'); plt.xscale('log');
        plt.axvline(x=maxpowfreq,color='r',ls=':')
        plt.plot(freq[minfloc:maxfloc],power[minfloc:maxfloc],lw
            =1)
        plt.title(r'Power spectrum of $\theta(t)$')
        plt.xlabel(r'frequency ($Hz$)'); plt.ylabel('Power (
            arbitrary units)')

    def plot_run_power_spectrum(self,sig,wid=129,minfrq=None,
        maxfrq=None):
        runavgsig = self.get_run_average(sig,wid)
        power,omega = self.get_power(self.bratlett_window(
            runavgsig)); freq = omega/(2*np.pi)
        minfloc = np.searchsorted(freq,minfrq)+1 if minfrq != None
             else 0
        maxfloc = np.searchsorted(freq,maxfrq)+1 if maxfrq != None
             else len(omega)
        maxpowloc = power.argmax(); maxpowfreq= freq[maxpowloc]

        plt.subplot(self.spl+self.pcnt); self.pcnt += 1; plt.
            tight_layout()
        plt.yscale('log'); plt.xscale('log');
        plt.axvline(x=maxpowfreq,color='r',ls=':')
        plt.plot(freq[minfloc:maxfloc],power[minfloc:maxfloc],lw
            =1)
        plt.title(r'Max power at $\omega$= {:2}, f={:2}'.format(
            maxpowfreq*2*np.pi,maxpowfreq))
```

```
        plt.title(r'Power spectrum of $\theta(t)$ with running
            average of width = {}'.format(wid))
        plt.xlabel(r'frequency ($Hz$)'); plt.ylabel('Power (
            arbitrary units)')

    def plot_sliced_power_spectrum(self,sig,slwid=16384,minfrq=
        None,maxfrq=None):
        tmin = self.t[0]; tmax = self.t[slwid]; dt = (tmax-tmin)/
            slwid;
        dw = 2*np.pi/(slwid*dt); omega = np.arange(slwid)*dw; freq
            =omega/(2*np.pi)

        N = len(sig); padlength = slwid - N % slwid
        longsig = np.concatenate((sig,np.zeros(padlength)),axis=0)
            ;

        minfloc = np.searchsorted(freq,minfrq)+1 if minfrq != None
             else 0
        maxfloc = np.searchsorted(freq,maxfrq)+1 if maxfrq != None
             else len(omega)

        cnt = 0; avpwr  = 0
        for pos in range(0,len(longsig),slwid):
            piece = longsig[pos:pos+slwid]; cnt+=1
            power,omg = self.get_power(self.bratlett_window(piece)
                );
            avpwr += power

        avpwr /= cnt

        maxpowloc = avpwr.argmax(); maxpowfreq= freq[maxpowloc]
        print('max power freq',maxpowfreq)

        plt.subplot(self.spl+self.pcnt); self.pcnt += 1; plt.
            tight_layout()
        plt.yscale('log'); plt.xscale('log')
        plt.axvline(x=0.1061,color='r',ls=':')
        plt.plot(freq[minfloc:maxfloc],avpwr[minfloc:maxfloc])
        plt.title(r'Smoothed power spectrum of $\theta(t)$ with
            slice length = {}'.format(slwid))
        plt.xlabel(r'Frequency(Hz)'); plt.ylabel('Power (arbitrary
             units)');



if __name__ == '__main__':
    FP = ForcedPendulum('./files/pendulum2.dat')
    FP.spl = 110
    #FP.plot_theta(); #tikz_save('images/Pendulum.tex')
    #FP.plot_power_spectrum(FP.theta,minfrq=0.01,maxfrq=2)
    #FP.plot_run_power_spectrum(FP.theta,wid=129,minfrq=0.01,
        maxfrq=2)
    FP.plot_sliced_power_spectrum(FP.theta,minfrq=0.01,maxfrq=2)

    plt.show()
```

Question Four

```
#!/usr/bin/env python3

import itertools
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal


class NoiseReduction():
    def __init__(self,flname):
        self.filename = flname
        self.slc = 20
        self.spl = 210; self.pcnt = 1
        self.read_file()
        self.a = 2048
        self.trc = 10

    def read_file(self):
        content = np.genfromtxt(self.filename)
        self.j     = content[:,0];
        self.c     = content[:,1]
        self.N     = len(self.c)

    def get_threshold_loc(self,sig,thr):
        return self.trc

    def get_power(self,sig):
        sft = np.fft.fft(sig)
        power = 1/len(sft)*(np.abs(sft))**2
        return power

    def sys_function(self):
        hn = int(self.N/2)
        j = np.arange(hn+1)
        sysf = np.zeros(self.N)
        sysf[:hn+1] = np.exp(-j**2/self.a**2)
        sysf[-hn:] = sysf[hn:0:-1]
        sysf /= np.sum(sysf)
```

```
        return sysf

    def truncate_noise(self,sig):
        thr = 1e-4
        scp = np.fft.fft(sig)
        thl = self.get_threshold_loc(scp,thr)
        scp[thl:-thl] = 0
        return scp

    def divide_complex(self,Nr,Dr):
        N = len(Nr)
        nsig = np.zeros(N,dtype=complex)
        nsig = np.where(abs(Nr)>0,Nr/Dr,nsig)
        return nsig

    def get_back_signal(self,sig):
        csig = sig
        truncated = self.truncate_noise(csig)
        ftgaussian = np.fft.fft(self.sys_function())
        ftsig = self.divide_complex(truncated,ftgaussian)
        orgsig = np.fft.ifft(ftsig)
        return orgsig


    def get_back_noise(self,corrupt_sig):
        osig = self.get_back_signal(corrupt_sig)
        fftosig = np.fft.fft(osig)
        fftgaussian = np.fft.fft(self.sys_function())
        fftcorrupt_sig = np.fft.fft(corrupt_sig)

        fftnoise = fftcorrupt_sig/fftgaussian - fftosig
        noise = np.fft.ifft(fftnoise)
        #noise = np.fft.ifft(fftcorrupt_sig)
        plt.plot(noise)
        #return noise


    def plot_org_sig(self,sig):
        osig = self.get_back_signal(sig)
        plt.plot(osig)


    def plot_power_spectrum(self,sig):
        power = self.get_power(sig); lng = len(power)
        omega = 1/len(sig) *2*np.pi* np.arange(lng)
        plt.subplot(self.spl+self.pcnt); self.pcnt += 1
        #plt.yscale('log'); plt.xscale('log');
        plt.plot(omega,power)
        plt.subplot(self.spl+self.pcnt); self.pcnt += 1
        plt.yscale('log'); plt.xscale('log');
        plt.plot(power,lw=1)

if __name__ == '__main__':
    NR = NoiseReduction('./files/corrupt.dat')
    NR.spl = 210
    #NR.plot_power_spectrum(NR.c)
    NR.plot_org_sig(NR.c)
    #NR.get_back_noise(NR.c)
    #plt.plot(NR.c)

    plt.show()
```

Chapter 2

Galactic Astrophysics

2.1 Homework One

2.1.1. Assume that the Galaxy is 10Gyr old, the rate of star formation in the past was proportional to $e^{\frac{-t}{T}}$ where t is the time since the galaxy formed and $T = 3$Gyr, and the stellar lifetimes are given by

$$t(M) = 10Gyr\left(\frac{M}{M_\odot}\right)^{-3}$$

Calculate the framctions of all (a) $2M_\odot$ and (b) $5M_\odot$ stars ever formed that are still around today.
Solution:
Let t_2 and t_5 be the lifetimes of $2M_\odot$ stars and $5M_\odot$ stars. Then

$$t_2 = 10\ times\left(\frac{2M_\odot}{M_\odot}\right)^{-3} = 1\frac{1}{4}Gyr = 1.25Gyr$$

$$t_5 = 10\ times\left(\frac{5M_\odot}{M_\odot}\right)^{-3} = \frac{2}{25}Gyr = 0.8Gyr$$

If N_{2f} is the total $2M_\odot$ stars ever formed, then

$$N_{2f} = \int_0^{10} ke^{-\frac{t}{T}}dt = -\frac{k}{T}\left[e^{-\frac{10}{3}} - 1\right] = -0.32k$$

Any $2M_\odot$ star formed earlier than t_2 from today are all gone so the remaining $2M_\odot$ stars are formed between $10 - t_2 = 10 - 1.25 = 8.75$Gyr and today (10Gyr) from the beginning.

$$N_{2r} = \int_{8.75}^{10} ke^{-\frac{t}{T}}dt = -\frac{k}{T}\left[e^{-\frac{10}{3}} - e^{-\frac{8.75}{3}}\right] = -6.14\times10^{-3}k$$

So the ratio of total $2M_\odot$ star still formed to that are still around is

$$\frac{N_{2r}}{N_{2f}} = \frac{-6.14\times10^{-3}k}{-0.32k} = 1.91\times10^{-2}$$

Since the star formation rate is independent of mass, the total $5M_\odot$ stars ever formed is equal to the total $2M_\odot$ stars. So, $N_{5f} = -0.321k$. Any $5M_\odot$ star formed earlier than t_5 from today are all gone so the remaining $5M_\odot$ stars are formed between $10 - t_2 = 10 - 0.08 = 9.92$Gyr and today (10Gyr) from the beginning.

$$N_{5r} = \int_{9.92}^{10} ke^{-\frac{t}{T}}dt = -\frac{k}{T}\left[e^{\frac{10}{3}} - e^{-\frac{9.92}{3}}\right] = -3.21 \times 10^{-4}k$$

So the ratio of total $5M_\odot$ star still formed to that are still around is

$$\frac{N_{5r}}{N_{5f}} = \frac{-3.21 \times 10^{-4}k}{-0.32k} = 9.99 \times 10^{-4}$$

□

2.1.2. (a) A close (i.e. unresolved) binary consists of two stars each of apparent magnitude m. What is the apparent magnitude of the binary?
(b) A star has apparent magnitude $m_V = 10$ and is determined spectroscopically to be an A0 main sequence star. What is its distance? (See Sparke & Gallagher Table 1.4.)
Solution:
The flux(f) magnitude(m) relation is $m = -2.5\log(f)$. So the flux of each stars is given by.

$$f = 10^{-\frac{m}{2.5}}$$

The flux is additive so the total flux of binary is just twice of this $f_{tot} = 2 \times f = 2 \times 10^{-\frac{m}{2.5}}$. Now the apparant magnitude (m) of the binary is:

$$m = -2.5\log(f_{tot}) = -2.5\log\left(2 \times 10^{-\frac{m}{2.5}}\right) = -2.5\left(\log(2) - \frac{m}{2.5}\right) = m - 0.75$$

So the apparant magnitude of binary is $m - 0.75$.
Given that the apparant magnitude of the star is $m_V = 10$, As it is a A_0 from the table the value for absolute magnitude is found to be $M_V = 0.80$. We know that the relation between the absolute magnitude(M_V) and apparant magnitude(m_V) and the distane of the star (r),

$$\begin{aligned} M_V - m_v &= 5(1 - \log(r)) \qquad \text{where r is in parsec} \\ -9.2 &= 5(1 - \log(r)) \\ \log(r) &= 2.84 \\ r &= 10^{2.84} = 691.83pc \end{aligned}$$

So the distane of the star is 691.83pc □

2.1.3. If the mass function for stars follows the Salpeter distribution, with

$$\xi(M)\frac{dN}{dM} = AM^{2.35}$$

(where dN is the number of stars with masses between M and M + dM; see Sparke & Gallagher, p. 66), for $M_l < M < M_u$, with Ml Mu, and the stellar mass–luminosity relation is

$$L(M) \propto M^4,$$

show that the total number and total mass of stars depend mainly on Ml , while the total luminosity depends mainly on Mu. Specifically, for Ml = $0.2M_\odot$ and Mu = $100M_\odot$, calculate the masses M1 and M2 such that 50% of the total mass is contained in stars with $M < M1$, while 50% of the total luminosity is contained in stars with $M > M2$.
Solution:
The total mass of dN stars is MdN. So the total mass of the range is

$$M_{tot} = \int_{M_l}^{M_u} MdN = \int_{M_l}^{M_u} M \cdot AM^{-2.35}dN = A\int_{M_l}^{M_u} M^{-1.35}dN = \frac{A}{-0.35}\left(M_u^{-0.35} - M_l^{-0.35}\right)$$

Since we have to find $M_u = M_1$ such that half the total mass between $0.2M_\odot$ and $100M_\odot$ is to be equal to the total mass in the range $0.2M_\odot$ and M_1. Let's suppose $M_1 = \alpha M_\odot$. So,

$$\frac{1}{2}\left[\frac{A}{-0.35}\left\{(100M_\odot)^{-0.35} - (0.2M_\odot)^{-0.35}\right\}\right] = \left[\frac{A}{-0.35}\left\{(\alpha M_\odot)^{-0.35} - (0.2M_\odot)^{-0.35}\right\}\right]$$

$$\frac{1}{2}\left[100^{-0.35} - 0.2^{-0.35}\right] = \left[\alpha^{-0.35} - 0.2^{-0.35}\right]$$

$$\alpha = \left[\frac{100^{-0.35} + 0.2^{-0.35}}{2}\right]^{\frac{1}{-0.35}}$$

$$\alpha = 1.06$$

So for the star in the range $0.2M_\odot$ to $1.06M_\odot$ have half the total number of the stars. The luminosity of each star of mass M is proportional to M^4 and there are dN such stars. So the total luminosity of starts between mass M and $M + dM$ is proportional to $M^4 dN$, So the total luminosity of the range M_l and M_u is a constant times

$$L_{tot} = \int_{M_l}^{M_u} M^4 dN = \int_{M_l}^{M_u} M^4 \cdot AM^{-2.35} dN = A\int_{M_l}^{M_u} M^{1.65} dN = \frac{A}{2.65}\left(M_u^{2.65} - M_l^{2.65}\right)$$

Since we have to find $M_l = M_2$ such that half the total luminosity between $0.2M_\odot$ and $100M_\odot$ is to be equal to the total luminosity in the range M_l and $100M_\odot$. Let's suppose $M_1 = \beta M_\odot$. So,

$$\frac{1}{2}\left[\frac{A}{2.65}\left\{(100M_\odot)^{2.65} - (0.2M_\odot)^{2.65}\right\}\right] = \left[\frac{A}{2.65}\left\{(100M_\odot)^{2.65} - (\beta M_\odot)^{2.65}\right\}\right]$$

$$\frac{1}{2}\left[100^{2.65} - 0.2^{2.65}\right] = \left[100^{2.65} - \beta^{2.65}\right]$$

$$\beta = \left[\frac{100^{2.65} + 0.2^{2.65}}{2}\right]^{\frac{1}{2.65}}$$

$$\beta = 76.98$$

$M_2 = 76.98M_\odot$ So the stars in the range $77M_\odot$ to $100M_\odot$ have half the luminosity as that of total stars in the range. □

2.1.4. Astronomers often approximate the stellar mass function (M) by a Salpeter power-law with a low-mass cutoff, but the Kroupa distribution

$$\xi(M) = \begin{cases} CM^{-0.3} & \text{for } M \leq 0.1M_\odot \\ BM^{-1.3} & \text{for } 0.1M_\odot < M \leq 0.5M_\odot \\ AM^{-2.35} & \text{for } M > 0.5M_\odot \end{cases}$$

is actually a much better description [A is the same as in part (a) and the other constants B and C are chosen to ensure that is continuous.] If the upper mass limit in all cases is Mu = $100M_\odot$ and we assume the same simplified mass–luminosity relation as in part (a), what low-mass cutoff Ml must be chosen in order that the truncated power-law has the same (i) total number of stars, (ii) total mass, and (iii) total luminosity as the Kroupa distribution?

Solution:

Since the given function $\xi(M)$ should be contunuous, each piece should have equal value at the boundary.

$$B(0.5M_\odot)^{-1.3} = A(0.5M_\odot)^{-2.35} \Rightarrow B = 2.070M_\odot^{-1.05}A$$

$$C(0.1M_\odot)^{-0.3} = B(0.1M_\odot)^{-1.35} \Rightarrow C = 10M_\odot^{-1}B = 20.70M_\odot^{-2.05}A$$

The total number of stars given by Kroupa distribution is

$$N = \int_0^{100M_\odot} \xi(M)dM = \int_0^{0.1M_\odot} 20.70AM_\odot^{-2.05}M^{-0.3}dM + \int_{0.1M_\odot}^{0.5M_\odot} 2.07AM_\odot^{-1.05}M^{-1.3}dM + \int_{0.5M_\odot}^{100M_\odot} AM^{-2.35}dM$$

$$N = 5.90AM_\odot^{-1.35} + 5.27AM_\odot^{-1.35} + 1.88AM_\odot^{-1.35} = 13.05M_\odot^{-1.35}A$$

Also the total number of star given by salpeter distribution with lower mass limit as $(\alpha M_\odot)$

$$N = \int_{\alpha M_\odot}^{100M_\odot} AM^{-2.35}dM$$

$$= 0.74(\alpha M_\odot)^{-1.35}A - 0.0014M_\odot^{-1.35}A$$

Equating these values

$$13.05M_\odot^{-1.05}A = 0.74(\alpha M_\odot)^{-1.35}A - 0.0014M_\odot^{-1.35}A$$

$$\Rightarrow \alpha^{-1.35} = 17.63$$

$$\Rightarrow \alpha = 0.11$$

Therefore the lower limit is $0.11M_\odot$ if Salpeter distribution and Kroupa distribution have the same number of stars.

Working in the units of $M_\odot = 1$ and $A = 1$:
The total Mass of stars given by Kroupa distribution is

$$M = \int_0^{100} M\xi(M)dM = \int_0^{0.1} 20.70M^{0.7}dM + \int_{0.1}^{0.5} 2.07M^{-0.3}dM + \int_{0.5}^{100} M^{-1.35}dM$$

$$M = 0.24 + 1.23 + 3.07 = 4.54$$

Also the total Mass of star given by salpeter distribution with lower mass limit as $(\alpha M_\odot)$

$$M = \int_{\alpha}^{100} M \times M^{-2.35}dM$$

$$= 2.85\alpha^{-0.35} - 0.57$$

Equating these values

$$4.54 = 2.85\alpha^{-0.35} - 0.57$$

$$\Rightarrow \alpha^{-0.35} = 1.79$$

$$\Rightarrow \alpha = 0.19$$

Therefore the lower limit is $0.19M_\odot$ for the Salpeter distribution and Kroupa distribution to have the same total mass.

The total Luminosity of stars given by Kroupa distribution is

$$L = \int_0^{100} M^4\xi(M)dM = \int_0^{0.1} 20.70AM^{3.7}dM + \int_{0.1}^{0.5} 2.07M^{2.7}dM + \int_{0.5}^{100} M^{1.65}dM$$

$$L = 8.78 \times 10^{-5} + 0.042 + 75292.85 = 75292.89$$

Also the total Luminosity of all stars given by salpeter distribution with lower mass limit as $(\alpha M_\odot)$

$$L = \int_{\alpha}^{100} M^4 \times M^{-2.35} dM$$
$$= 75292.92 - 0.37\alpha^{2.65}$$

Equating these values

$$75292.89 = 75292.92 - 0.37\alpha^{2.65}$$
$$\Rightarrow \alpha^{2.65} = 0.07$$
$$\Rightarrow \alpha = 0.36$$

Therefore the lower limit is $0.36 M_\odot$ for Salpeter distribution and Kroupa distribution have the same Luminosity.

□

2.1.5. (a) Use Gauss's law to derive an expression for the gravitational force in the z direction due to an infinite sheet of surface density Σ lying in the x–y plane. (b) A star has velocity 30 km/s perpendicular to the Galactic plane as it crosses the plane, and is observed to have a maximum departure above the plane of 500 pc. Approximating the disk as an infinite gravitating sheet of matter, estimate its surface density Σ (i) in kgm^2 and (ii) in $M_\odot pc^{-2}$

Solution:

The gravitational flux(Φ) thourgh a closed surface enclosing mass M_{encl} is

$$\Phi = 4\pi G M_{encl} \tag{2.1}$$

If we assume the galactic plane as an infinite sheet of mass uniformly distributed over a surface with surface density Σ and we take the Gaussian surface as a cylynder of radius a perpendicular to the plane, then the total mass included within the cylinder would be $M_{encl} = \text{Area} \times \Sigma = \pi a^2 \Sigma$. But the total surface area of cylinder that is perpendicular(z direction) to the Plane is $2\pi a^2$. If E is the Gravitational field at the cylinder surface, then total flux (Φ) through the area is $E \times 2\pi a^2$ Substuting the values of Φ and M_{encl} in (2.1) we get.

$$2\pi a^2 E = 4\pi G(\pi a^2 \Sigma)$$
$$\Rightarrow E = 2\pi G \Sigma$$

So the gravitatational force per unit mass in the z direction is $2\pi G\Sigma$.

Given that a star with velocity $v = 30 km/s$ and travels a max distance of $s = 500pc = 1.543 \times 10^{19} m$. Sine the gravitational field is constant and is independent of distance above the galactic plane. We can use the constant accleration kinematics relation $v_f^2 - v_i^2 = 2as$. Since the speed at maximuh distance is zero.

$$a = \frac{v_i^2}{2s}$$

But the accleration $a = 2\pi G\Sigma$

$$\Sigma = \frac{v_i^2}{4\pi G s} = \frac{(3 \times 10^4)^2}{2 \times 1.543 \times 10^{19} \times 4\pi \times 6.672 \times 10^{-11}} = 0.069 kgm^{-2}$$

Since $1kg = 5.02 \times 10^{-31} M_\odot$ and $1m^{-2} = 9.52 \times 10^{32} pc^{-2}$

$$\Sigma = 0.069 \times 5.02 \times 10^{-31} \times 9.52 \times 10^{32} M_\odot pc^{-2} = 33.30 M_\odot pc^{-2}$$

So the surface mass density Σ for the given planar galaxy is $0.069 kgm^2 \equiv 33.30 M_\odot pc^{-2}$. □

2.2 Homework Two

2.2.1. A certain telescope has limiting visual apparent magnitude $m_V = 22$. What is the maximum distance at which it could detect (a) the Sun (absolute magnitude $M_V = 4.8$), (b) an RR Lyrae variable ($M_V = 0.75$), a Cepheid variable ($M_V = -3.5$), and (d) a type Ia supernova ($M_V = -20$).
Solution:
If m_V is the limiting aparrant magnitude of the telescope, anything with apparent magnitude greater than m_V would not be resolved by the telescope. So the maximum distance that the telescope can still resolve is the distance in which the apparent magnitude of each of the stars is equal to the limiting apparant magnitude.

If we suppose d_{max} is the maximum distance. Then

$$M_v - m_v = -5\log\left(\frac{d_{max}}{10}\right)$$

$$Rightarrow d_{max} = 10 \times 10^{\frac{m_v - M_v}{5}}$$

$$\text{Since the limiting magnitude } (m_v) = 22$$

$$d_{max} = 10 \times 10^{\frac{22-M_v}{5}} \text{in Parsec}$$

- For Sun $M_V = 4.8$, limiting distance $d_{max} = 10 \times 10^{\frac{22-4.8}{5}} = 27.54kpc$
- For RR Lyre $M_V = 0.75$, limiting distance $d_{max} = 10 \times 10^{\frac{22-0.75}{5}} = 177.82kpc$
- For Cepheid variable $M_V = -3.5$, limiting distance $d_{max} = 10 \times 10^{\frac{22+3.5}{5}} = 1.25Mpc$
- For Ia Supernova $M_V = -20$, limiting distance $d_{max} = 10 \times 10^{\frac{22+20}{5}} = 2.511Gpc$

□

2.2.2. A simple axisymmetric model of the stellar number density $n(R, z)$ in the Galactic disk is

$$n(R, z) = n_0 e^{-R/h_R} e^{-|z|/h_z},$$

where R is distance from the Galactic center, z is distance from the disk plane, and h_R and h_z are (constant) scale heights. (a) If all stars have the same luminosity L_*, integrate the above expression with respect to z to determine the disk surface brightness $\Sigma(R)$ (that is, the total luminosity per unit area at any given location). (b) Now integrate Σ with respect to R to determine the total luminosity L_G of the Galaxy. (c) If $L_G = 2 \times 10^{10} L_\odot$, and $h_R = 4$ kpc, what is the local surface brightness in the vicinity of the Sun, at R = 8 kpc? (d) If $h_z = 250$ pc and $L_* = L_\odot$, calculate the local density of stars in the solar neighborhood (at z = 0).
Solution:
Given all stars have same luminosity L_* the luminosity per unit area is:

$$\Sigma(R) = \int_{-\infty}^{\infty} L_* n_0 e^{-\frac{R}{h_R}} e^{-\frac{|z|}{h_z}} dz = 2\int_{0}^{\infty} L_* n_0 e^{-\frac{R}{h_R}} e^{-\frac{z}{h_z}} dz$$

$$= -L_* h_z n_0 e^{-\frac{R}{h_R}} \left[e^{-\frac{z}{h_z}}\right]_0^{\infty} = 2n_0 h_z e^{-\frac{R}{h_R}} L_*$$

Now for the total Luminosity the function $\Sigma(R)$ is integrated from $R = 0$ to ∞.

$$L_G = \int_0^{\infty} \Sigma(R) dR = \int_0^{\infty} 2n_0 h_z e^{-\frac{R}{h_R}} L_* dR$$

$$= 2n_0 h_z L_* \left(-h_r \left[e^{-\frac{R}{h_R}}\right]_0^{\infty}\right) = 2n_0 h_R h_z L_*$$

The above expression for L_G gives the total luminosity of galaxy in terms of the luminosity of each stars L_*

For vicinity of sun at $R = 8kpc$ and $L_G = 2 \times 10^{10} L_\odot$ and $h_R = 4kpc$

$$L_G = 2n_0 h_R h_z L_* \qquad \Rightarrow h_z = \frac{L_G}{2L_* h_R n_0} \Rightarrow h_z = \frac{2.5 \times 10^9}{n_0 L_*} L_\odot$$

So the local surface brightness $\Sigma(R)$ at the vicinity of sun then is

$$\Sigma(R) = 2n_0 h_z e^{-\frac{R}{h_R}} L_* = 2n_0 \cdot \frac{2.5 \times 10^9}{n_0 L_*} \cdot e^{-\frac{8}{4}} L_* \Rightarrow 6.76 \times 10^8 \ L_\odot$$

The local density of stars around $z = 0$ is

$$n(8kpc, 0) = n_0 e^{\frac{8}{4}} e^0 = 0.13 n_0$$

□

2.2.3. (a) Given the definitions of the Oort constants A and B presented in class (Eqs. 2.13 and 2.16 in the text),

$$A = -\frac{1}{2} R \left(\frac{V}{R}\right)' \Bigg|_{R=R_0} \qquad B = -\frac{1}{2} \frac{(RV)'}{R} \Bigg|_{R=R_0}$$

verify that $A + B = V'(R_0)$ and $AB = V_0/R_0$, where V (R) is the Galactic rotation law, R_0 is the distance from the Sun to the Galactic center, and $V_0 = V(R0)$.
(b) Hence write down an estimate of V_0, if $R_0 = 8$kpc.
(c) Consider the spherically symmetric density distribution given by

$$\rho(R) = \rho_0 \left(1 + \frac{R^2}{a^2}\right)^{-1}$$

Derive an expression for the mass inside radius R. What is the circular orbital speed V (R) at radius R? Hence determine the form of A(R) and B(R) for $R \gg a$.
Solution:

$$\begin{aligned} A &= -\frac{1}{2} R \left(\frac{V}{R}\right)' \\ &= -\frac{1}{2} R \left(\frac{V'}{R} - \frac{V}{R^2}\right) \\ &= -\frac{1}{2} V' + \frac{1}{2}\frac{V}{R} \end{aligned}$$

Evaluating at $R = R_0$

$$A = -\frac{1}{2} V'(R_0) + \frac{1}{2}\frac{V(R_0)}{R_0}$$

$$\begin{aligned} B &= -\frac{1}{2}\frac{(RV)'}{R} \\ &= -\frac{1}{2}\frac{1}{R}(V + RV') \\ &= -\frac{1}{2}\frac{V}{R} - \frac{1}{2} V' \end{aligned}$$

Evaluating at $R = R_0$

$$B = -\frac{1}{2}\frac{V(R_0)}{R_0} - \frac{1}{2} V(R_0)'$$

Now that we have the values for each constants A and B.

$$A + B = -V'(R_0) \qquad A - B = \frac{V(R_0)}{R_0}$$

Let us conider a hollow shell of radius R with thickness dR. Then the volume of the differential shell is

$$dV = 4\pi R^2 dR$$

The differential relation for mass can be written as.

$$\begin{aligned} dM &= \rho(R)dV \\ &= \rho_0 \left(1 + \frac{R^2}{a^2}\right)^{-1} 4\pi R^2 dR \end{aligned}$$

The total mass enclosed in the sphere of radius R is given by the integral of dM from 0 to R

$$\begin{aligned} M(R) = \int_0^R dM &= \int_0^R \rho_0 \left(1 + \frac{R^2}{a^2}\right)^{-1} 4\pi R^2 dR \\ &= 4\pi\rho_0 \int_0^R \frac{R^2}{1 + \frac{R^2}{a^2}} dR \\ &= 4\pi\rho_0 a^2 \left(R - a \tan^{-1}\left({}^{R}\!/_{a}\right)\right) \end{aligned}$$

$$M(R) = 4\pi\rho_0 a^2 \left(R - a \tan^{-1}\left({}^{R}\!/_{a}\right)\right) \tag{2.2}$$

To calculate the $V(R)$ we can use the relation.

$$\begin{aligned} \frac{V^2(R)}{R} &= \frac{GM(R)}{R^2} \\ \frac{V^2(R)}{R} &= \frac{G4\pi\rho_0 a^2 \left(R - a\tan^{-1}\left({}^{R}\!/_{a}\right)\right)}{R^2} \qquad \text{Substuting M(R) from (2.2)} \\ V(R) &= 2a\sqrt{G\pi\rho_0\left(1 - \frac{a}{R}\tan^{-1}\left(\frac{R}{a}\right)\right)} \end{aligned}$$

if $R \gg a$ then $\tan^{-1}\left(\frac{R}{a}\right) \approx \frac{\pi}{2}$ also $\frac{a}{R} \to 0$ Then.

$$V(R) = 2a\sqrt{G\pi\rho_0}$$

Since V has no dependence on R, $V' = 0$

$$A(R) = -\frac{1}{2}V' + \frac{1}{2}\frac{V}{R} = 0 + \frac{1}{2}\frac{2a\sqrt{G\pi\rho_0}}{R} = \frac{a\sqrt{G\pi\rho_0}}{R}$$

$$B(R) = -\frac{1}{2}\frac{V}{R} - \frac{1}{2}V' = -\frac{1}{2}\frac{2a\sqrt{G\pi\rho_0}}{R} + 0 = -\frac{a\sqrt{G\pi\rho_0}}{R}$$

□

2.2.4. If our Galaxy has a flat rotation curve with $V_0 = 210$ km/s and the total luminosity of the disk is as in Problem 2, what is the Galactic mass to light ratio M/L inside (a) the solar circle ($R_0 = 8$ kpc), (b) $10R_0$? Compare these with the mass to light ratio of a Salpeter stellar mass distribution (see Homework 1, Problem 3) with $M_l = 0.2M_\odot$, $M_u = 100M_\odot$.
Solution:
Total luminosity inside of radius R can be calculated as

$$\begin{aligned} L(R) = \int_0^R \Sigma(R)dR &= \int_0^R 2n_0 h_z e^{-\frac{R}{h_R}} L_* \\ &= 2h_R h_z n_0 \left(1 - e^{-\frac{R}{h_R}} L_*\right) \end{aligned}$$

Substuting $h_r = 4kpc, h_z = 250pc, L_* = L_\odot$ in above expression

$$L(R) = 2 \times 10^6 \left(1 - e^{\frac{R}{4kpc}}\right) n_0 L_\odot$$

If the rotation curve is flat, the mass can be calculated as

$$M(R) = \frac{RV^2}{G} = 48.83R\, M_\odot/pc$$

For $R = 8kpc$

$$L(R) = 2.0 \times 10^6 \left(1 - e^{-2}\right) n_o L_\odot = 1.72 \times 10^6 n_0 L_\odot$$

$$M(R) = \frac{RV^2}{G} = 48.83 \times 8000 M_\odot = 3.90 \times 10^5 M_\odot$$

The ratio then is:

$$M/L = \frac{1.72 \times 10^6 n_0 L_\odot}{3.90 \times 10^5 M_\odot} = 0.22 n_0 {}^{M_\odot}\!/_{L_\odot}$$

For $R = 10R_0 = 80kpc$

$$L(R) = 2.0 \times 10^6 \left(1 - e^{-20}\right) n_o L_\odot = 1.99 n_0 L_\odot$$

$$M(R) = \frac{RV^2}{G} = 48.83 \times 80000 M_\odot = 3.90 \times 10^6 M_\odot$$

The ratio then is:

$$M/L = \frac{1.99 L_\odot}{3.90 \times 10^6 M_\odot} = 5.1 \times 10^{-7} n_0 {}^{M_\odot}\!/_{L_\odot}$$

For salpeter distribution $\xi(M) = AM^{-2.35}$ The total mass is

$$M = \int_{.2M_\odot}^{100M_\odot} M\xi(M)dM = \int_{.2M_\odot}^{100M_\odot} AM^{-1.35}dM = 1.49 \times 10^6 A$$

$$L = \int_{.2M_\odot}^{100M_\odot} M^4\xi(M)dM = \int_{.2M_\odot}^{100M_\odot} AM^{1.65}dM = 2.13 \times 10^4 A$$

The ratio is

$$M/L = \frac{1.49 \times 10^6 A}{2.13 \times 10^4 A} = 69.95$$

□

2.3 Homework Three

2.3.1. Neutral hydrogen atoms in the cool interstellar medium have number density $n_H \approx 1cm^{-3}$ and temperature T100 K.

(a) Show that the average speed $\bar{v}$ of these atoms, defined by $\frac{1}{2}m_H\bar{v}^2 = \frac{3}{2}kT$(where m_H is the mass of hydrogen atom and k is Boltzmann's constant), is

$$\bar{v} \approx 2kms^{-1}\left(\frac{T}{100K}\right)^{1/2}.$$

Solution:

Given that the average speed $\bar{v}$ of these atoms, defined by $\frac{1}{2}m_H\bar{v}^2 = \frac{3}{2}kT$ It can be rearranged into

$$\bar{v} = \sqrt{\frac{3kT}{m_H}} = \sqrt{\frac{3 \times k \times 100}{m_H}}\sqrt{\frac{T}{100K}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 100}{1.67 \times 10^{-27}}}\left(\frac{T}{100K}\right)^{1/2} = 1.57kms^{-1}\left(\frac{T}{100K}\right)^{1/2}$$

□

(b) Hence show that the typical atomic center-of-mass kinetic energy is much greater than the energy difference between the hyperfine states associated with the 21-cm radio line.

The mean time between collisions for atoms in this environment is a few thousand years, while the mean time for an excited atom to emit a $21cm$ photon is $\approx 1.1 \times 10^7$. As a result, the populations of the lower and upper hyperfine states are determined entirely by collisional processes and the states are populated proportional to their statistical weights, so three-quarters of all hydrogen atoms are in the upper state.

Solution:
The energy associated with 21cm line is

$$E = \frac{hc}{\lambda} = 9.485 \times 10^{-25}J = 5.92 \times 10^{-6}eV$$

The typical energy is 13.6eV which is much greater than the energy associated with 21cm line □

(c) Calculate the total 21-cm luminosity of a galaxy containing a total of $5 \times 10^9 M_\odot$ of neutral hydrogen.
Solution:
The total number of neutral hydrogen is $N = \frac{5 \times 10^9 M_\odot}{m_H} = 1.13 \times 10^{57}$.
The rate of emission of photon is $f = (1.1 \times 10^7)^{-1}/yr = 2.28 \times 10^{-15}s^{-1}$.
So the total luminosity due to 21cm photon is given by

$$L = N \cdot f \cdot E = 3.11 \times 10^{18}W$$

So the total luminosity fo the given galaxy is 3.11×10^{18} □

2.3.2. What are the sound speed and Jeans mass. (In all cases, assume an adiabatic index $\gamma = \frac{5}{3}$)

(a) in a molecular cloud core (pure H_2) of temperature 10 K and number density 1×10^6 molecules/cm^3?

Solution:
Mass of hydrogen molecule H_2 is $m_{H_2} = 3.34 \times 10^{-27}$, $T = 10K$

$$C_s = \sqrt{\frac{\gamma kT}{m}} = 262.41 m/s; \quad \rho = m_{H_2} n = 3.34 \times 10^{-15} kgm^{-3}$$
$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 8.048 \times 10^9 m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 7.29 \times 10^{15} kg$$

□

(b) in atomic hydrogen gas with temperature 100 K and number density 1 atom/cm^3
Solution:
Mass of hydrogen atom is $m_H = 1.67 \times 10^{-27}$, $T = 100K$

$$C_s = \sqrt{\frac{\gamma kT}{m}} = 117.56 m/s; \quad \rho = m_H n = 1.67 \times 10^{-21} kgm^{-3}$$
$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 5.09 \times 10^{13} m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 9.22 \times 10^{20} kg$$

□

(c) in hot ionized hydrogen with temperature 1×10^6 K and number density 1×10^{-3} protons/cm^3?
Solution:
Mass of ionized hydrogen molecule is $m_p = 1.67 \times 10^{-27}$, $T = 100K$

$$C_s = \sqrt{\frac{\gamma kT}{m}} = 1.17 \times 10^5 m/s; \quad \rho = m_p n = 1.67 \times 10^{-24} kgm^{-3}$$
$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 1.67 \times 10^{17} m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 2.91 \times 10^{28} kg$$

□

2.3.3. Air at sea level on Earth has density = 1.2 kg/m3 and sound speed vs = 330 m/s.

(a) What is its Jeans length? What is the Jeans mass?
Solution:

$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 5.339 \times 10^2 m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 7.65 \times 10^8 kg$$

The Jeans length is $533.9m$ and the Jeans mass is $7.65 \times 10^8 kg$ □

(b) By how much does the self-gravity of air change the frequency of a sound wave of wavelength 1 m?
Solution:
The frequency of $1m$ wavelength wave on earth is $f = v_s/\lambda = 330Hz$ The change in frequency due to gravitation is related by

$$f^2 - f_n^2 = \frac{G\rho}{\pi} \tag{2.3}$$

If we supppose changed frequency $f_n = f + \Delta f$ and Δf is very small then

$$f^2 - f_n^2 = f^2 - (f + \Delta f)^2 = f^2 - f^2\left(1 + \frac{\Delta f}{f}\right)^2 \approx f^2 - f^2\left(1 + 2\frac{\Delta f}{f}\right) = 2\Delta f \cdot f$$

Substuting this difference into (2.3) we get

$$\Delta f = \frac{G\rho}{2\pi f} = \frac{6.672 \times 10^{-11} \cdot 1.2}{2\pi \cdot 330} = 3.86 \times 10^{-14} Hz$$

□

2.3.4. (a) A particle is dropped (from radius a with zero velocity) into the gravitational potential corresponding to a static homogeneous sphere of radius a and density . Calculate how long the particle takes to reach the center of the sphere.

Solution:

Let the density of the mass density of the homogenous sphere be ρ. Also le the mass of the sphere within the shell of radius r be $M(r)$.

$$M(r) = \rho V(r) = \rho \frac{4}{3}\pi r^3$$

Writing the equation of motion from Newton's laws.

$$\ddot{r} = -\frac{GM(r)}{r^2} = -\frac{G\rho\frac{4}{3}\pi r^3}{r^2} = -\frac{4}{3}G\pi\rho r = -\omega^2 r \quad \left(\text{where } w^2 = \frac{4}{3}G\pi\rho\right) \tag{2.4}$$

The second order differential equation (2.4) is the well known SHM equation which has periodic solution of the form.

$$r(t) = A\cos(\omega t) + B\sin(\omega t) \tag{2.5}$$

Differentiating (2.5) we get

$$\dot{r}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

where A and B are the parameters determined by the boundary value. Since the particle starts from the surface of the sphere $r(0) = a$ and the initial speed $\dot{r}(0) = 0$. Using these boundary values we find the values of A and B. The thus determined are $A = a$ and $B = 0$. So (2.5) becomes

$$r(t) = a\cos(\omega t) \qquad \text{Where } \omega = \sqrt{{}^4\!/_3 G\pi\rho} \tag{2.6}$$

If T is the time the particle takes to reach the center of the spherical distribution then $r(T) = 0$ so the solution of (2.6) gives

$$\omega T = \frac{\pi}{2} \Rightarrow T = \frac{\pi}{2\omega} = \sqrt{\frac{3\pi}{16G\rho}}$$

The time T is the time the particle takes to reach the center of spherical distribution. □

(b) Calculate the time required for a homogeneous sphere of radius a and density with no internal pressure support to collapse under its own gravity.

Solution:

If the spherical distribution collapses by its own gravity, then as the particle on the surface is pulled inward towards the center, the mass compresses and so the mass inside the spherical shell at any time is constant

$$M(r) = \rho\frac{4}{3}\pi a^3$$

Writing the equation of motion from Newton's laws.

$$\ddot{r} = -\frac{GM(r)}{r^2} = -\frac{G\rho\frac{4}{3}\pi a^3}{r^2} = -\frac{\omega^2}{r^2} \quad \left(\text{where } w^2 = \frac{4}{3}G\pi\rho a^3\right) \tag{2.7}$$

We can transform $\ddot{r}$

$$\ddot{r} \equiv \frac{d\dot{r}}{dt} \equiv \frac{dr}{dt}\frac{d\dot{r}}{dr} \equiv \dot{r}\frac{d\dot{r}}{dr} \tag{2.8}$$

On using (2.8) (2.7) becomes

$$\dot{r}d\dot{r} = \left(-\frac{\omega^2}{r^2}\right)dr \qquad \Rightarrow \qquad \frac{1}{2}\dot{r}^2 = \frac{\omega^2}{r} + K$$

The boundary condition is that at $r = a$ the starting speed of particle is $\dot{r} = 0$ Substuting this back we find $K = {}^{-\omega^2}/_a$. We get

$$\dot{r} = \sqrt{2}\omega\sqrt{\frac{1}{r} - \frac{1}{a}} \qquad \Rightarrow \qquad \left(\frac{1}{r} - \frac{1}{a}\right)^{-\frac{1}{2}} dr = \sqrt{2}\omega dt \tag{2.9}$$

The solution of (2.9) is[1]

$$a^{\frac{3}{2}}\sin^{-1}\left(\sqrt{\frac{r}{a}}\right) - \frac{r-a}{\sqrt{\frac{1}{a} - \frac{1}{r}}} = \sqrt{2}\omega t + C \tag{2.10}$$

$$\lim_{r\to a}\frac{r-a}{\sqrt{\frac{1}{a} - \frac{1}{r}}} = 0; \qquad \lim_{r\to a}\sin^{-1}\left(\sqrt{\frac{r}{a}}\right) = \frac{\pi}{2} \qquad \Rightarrow \qquad C = \frac{\pi}{2}a^{3/2} \tag{2.11}$$

Using (2.11) in (2.10) we get

$$a^{\frac{3}{2}}\left(\sin^{-1}\left(\sqrt{\frac{r}{a}}\right) - \frac{\pi}{2}\right) - \frac{r-a}{\sqrt{\frac{1}{a} - \frac{1}{r}}} = \sqrt{2}\omega t \tag{2.12}$$

If T is the time the particle takes to reach the center $r(T) = 0$ so the solution of (2.10) gives

$$\sqrt{2}\omega T = \frac{\pi}{2}a^{\frac{3}{2}} \Rightarrow T = \frac{\pi}{2\sqrt{2}\omega}a^{3/2} = \sqrt{\frac{a^3\pi^2}{8\cdot {}^4\!/_3 G\rho\pi a^3}} = \sqrt{\frac{3\pi}{32G\rho}}$$

The time T is the time the particle takes to reach the center of spherical distribution which is the time of the collapse of the mass distribution under its own gravitational pull. □

2.4 Homework Four

2.4.1. Estimate the masses of star clusters having

(a) root mean square velocity 10 km/s and half-mass radius 10 pc,
Solution:
Given $v_{rms} = 10km/s$, the mean square speed is $\langle v^2\rangle = (10km/s)^2 = 1\times10^4$ The total mass is given by

$$M = \frac{6R_h\langle v^2\rangle}{G} = \frac{6\cdot1\times10^4\cdot10\times3.08\times10^{16}}{6.67\times10^{-11}} = 2.78\times10^{36}kg = 1.39\times10^6 M_\odot$$

So the mass of the cluster is $1.39\times10^6 M_\odot$ □

[1]solved by Sympy 1.1.1 under python 3.5

(b) mean density $100pc^{-3}$, rms velocity $2km/s$, and mean stellar mass $0.8M_\odot$,
Solution:
If the number density is n and average stellar mass is $\bar{m}$ then the mean mass density

$$\rho = n \cdot \bar{m} = 100pc^{-3} \cdot 0.8M_\odot = 80M_\odot/pc^3; v_{rms} = 2km/s \Rightarrow \left\langle v^2 \right\rangle = 4 \times 10^4$$

. The density volume relation $\rho = \frac{3M}{4\pi R^3} \Rightarrow R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$.

$$M = \frac{6R\left\langle v^2 \right\rangle}{G} = \frac{6\left\langle v^2 \right\rangle}{G}\left(\frac{3M}{4\pi\rho}\right)^{1/3} \Rightarrow M = \left(\frac{6\left\langle v^2 \right\rangle}{G}\left(\frac{3}{4\pi\rho}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}} = 4.53 \times 10^{34}kg = 2.27 \times 10^4 M_\odot$$

□

(c) dynamical time 1×10^6yr and radius 1 pc.
Solution:
The dynamical time $\tau = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}$. Using $\rho = \frac{3M}{4\pi R^3}$ we get

$$M = \frac{4\pi^2 R^3}{G\tau^2} = 1.75 \times 10^{35}kg = 8.79 \times 10^4 M_\odot$$

□

2.4.2. Interstellar gas in many galaxies is in virial equilibrium with the stars, in that the rms speed of the gas particles is the same as the rms stellar speed. Consider a large elliptical galaxy with a virial radius of 100 kpc and a mass of $1 \times 10^{12} M_\odot$ solar masses. Calculate the rms stellar velocity using the virial theorem. Hence estimate the temperature of the interstellar gas, assuming that it is composed entirely of hydrogen.
Solution:

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \left(\frac{GM}{6R}\right)^{\frac{1}{2}} = \left(\frac{6.67 \times 10^{-11} \cdot 1 \times 10^{12} \cdot 1.9 \times 10^{30}}{6 \cdot 1 \times 10^4 \cdot 3.08 \times 10^{16}}\right)^{\frac{1}{2}} = 2.68 \times 10^5 m/s = 268km/s$$

The mass of hydrogen is $m_H = 1.67 \times 10^{-27}kg$. If all the interstellar mass was composed of hydrogen then the temperature would be given by reation

$$\frac{1}{2}m_H \left\langle v^2 \right\rangle = \frac{3}{2}kT \Rightarrow T = \frac{m_H \left\langle v^2 \right\rangle}{3k} = \frac{1.67 \times 10^{-23} \cdot (268 \times 10^3)^2}{3 \cdot 1.68 \times 10^{23}} = 2.86 \times 10^6 K$$

□

2.4.3. Assuming an average stellar mass of $0.5M_\odot$ and $\Lambda = r_c/1AU$, lookup table values and find the relaxation time t_r at the center of globular cluster 47 Tucanae. Show that the crossing time $t_{cross} \approx 2r_c/\sigma_r \sim 1 \times 10^{-3} t_{relax}$
Solution:
The total number of stars in the cluster is given by

$$N = \frac{\text{Total Mass}}{\text{Mean Mass}} = \frac{800M_\odot}{0.5M_\odot} = 1600$$

The density of stars from table is $\rho = 10^{4.9} M_\odot/pc^3$. The dynamical time of the stars can be now calculated as

$$\tau = \left(\frac{GM}{r_c^3}\right)^{-\frac{1}{2}} = 3.09 \times 10^5 yr$$

.

Now the relaxation time

$$t_{relax} = \frac{N}{8.5\ln(\Lambda)}\tau = \frac{1600}{8.5\ln(r_c/1AU)}3.09\times10^5 = 4.89\times10^6 yr$$

The cross time is

$$\frac{t_{cross}}{t_{relax}} = \frac{2r_c}{\sigma_r t_{relax}} = \frac{2\cdot0.7pc}{1.1\times10^4\cdot4.89\times10^6} = 2.54\times10^{-2}$$

□

2.4.4. The velocities of stars in a stellar system are described by a three-dimensional Maxwellian distribution—that is,

$$f(v) = Av^2e^{-mv^2/2kT}$$

Here, A is a normalization constant, m is the stellar mass, assumed constant, k is Boltzmann's constant, and T is the temperature of the system. Verify the mean stellar kinetic energy is $\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT$

Solution:

The normalization condition gives

$$\int_0^\infty f(v)dv = \int_0^\infty Av^2e^{-mv^2/2kT}dv = 1$$

To carry out the integration lets make some change of variables

$$\frac{mv^2}{2kT} = x; \Rightarrow v = \sqrt{\frac{2kT}{m}x}; \quad dv = \frac{kT}{mv}dx, \qquad \text{As } v\to\{0,\infty\} \quad x\to\{0,\infty\}$$

Using these variable transformation, our normalization integral becomes.

$$A\int_0^\infty v^2e^{-x}\frac{kT}{mv}dx = A\int_0^\infty \frac{kT}{m}\sqrt{\frac{2kT}{m}}xe^{-x}dx = A\sqrt{2}\left(\frac{kT}{m}\right)^{\frac{3}{2}}\int_0^\infty \sqrt{x}e^{-x}dx = 1$$

But by definition of gamma function $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}$ we get. And $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

$$A\sqrt{2}\left(\frac{kT}{m}\right)^{\frac{3}{2}}\int_0^\infty x^{\frac{3}{2}-1}e^{-x}dx = A\sqrt{2}\left(\frac{kT}{m}\right)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right) = 1 \Rightarrow A = \frac{1}{\frac{1}{2}\sqrt{2\pi}\left(\frac{kT}{m}\right)^{\frac{3}{2}}}$$

The expectation value for the square of speed can be calculated as:

$$\langle v^2\rangle = \int_0^\infty v^2f(v)dv = A\int_0^\infty v^4e^{-mv^2/2kT}dv$$

Carrying out same transformations as above we get.

$$\begin{aligned}
\langle v^2 \rangle &= A \int_0^\infty 2^{\frac{3}{2}} \left(\frac{kT}{m}\right)^{\frac{5}{2}} x^{\frac{3}{2}} e^{-x} dx \\
&= A 2^{\frac{3}{2}} \left(\frac{kT}{m}\right)^{\frac{5}{2}} \int_0^\infty x^{\frac{5}{2}-1} e^{-x} = A 2^{\frac{3}{2}} \left(\frac{kT}{m}\right)^{\frac{5}{2}} \Gamma\left(\frac{5}{2}\right) = A 2^{\frac{3}{2}} \left(\frac{kT}{m}\right)^{\frac{5}{2}} \frac{3}{4}\sqrt{\pi} \\
&= \frac{1}{\frac{3}{2}\sqrt{2\pi}\left(\frac{kT}{m}\right)^{\frac{3}{2}}} \times 2^{\frac{3}{2}} \left(\frac{kT}{m}\right)^{\frac{5}{2}} \frac{3}{4}\sqrt{\pi} \\
&= \frac{3kT}{m} \\
\Rightarrow \frac{1}{2} m \langle v^2 \rangle &= \frac{3}{2} kT
\end{aligned}$$

So the kinetic energy of each mass is $\frac{3}{2}kT$ if the velocity destribution of the ensemble of mass follow Maxwellian distribution function. □

2.4.5. Work out the details of the simple evaporative model discussed in class. Stars evaporate from a cluster of mass M on a time scale $t_{ev} = \alpha t_R$, where $\alpha \gg 1$,so

$$\frac{dM}{dt} = -\frac{M}{\alpha t_R} \tag{2.13}$$

For pure evaporation, each escaping star carries off exactly zero energy (i.e. stars barely escape the cluster potential), so the total energy of the cluster remains constant.

(a) If the cluster potential energy can always be written as $U = -k\frac{GM^2}{2R}$ for fixed k, where R is a characterstic cluster radius, and assuming that the cluster is always in virial equilibrium, show that $R \propto M^2$ as the cluster evolves.

Solution:
The potential energy relation can be reorganized as

$$R = -\frac{kG}{2U}M^2; \qquad \Rightarrow R = \beta_0 M^2; \quad \Rightarrow R \propto M^2; \quad \text{Where } \beta_0 = -\frac{kG}{2U}$$

So $R \propto M^2$. □

(b) Assuming that the relaxation time t_R scales as $M^{1/2}R^{3/2}$ so

$$t_R = t_{R0}\left(\frac{M}{M_0}\right)^{1/2}\left(\frac{R}{R_0}\right)^{3/2} \tag{2.14}$$

Solve (2.13) to determine the lifetime of the cluster (in terms of its initial relaxation time t_{R0}). Also write down an expression for the mean cluster density as a function of time.
Solution:
We can write Eq. (2.14) as $t_R = \beta_1 M^{1/2}R^{3/2}$. Since $R = \beta_0 M^2$. We now have, $t_R = \beta_1 M^{1/2}(\beta_0 M^2)^{3/2}$;

$$\Rightarrow t_R = \beta_3 M^{7/2}$$

Suppose T is the lifetime of the cluster that had initial mass of M_i then as time goes from 0 to T mass goes from M_i to 0. Using t_R in Eq. (2.13) we get

$$\frac{dM}{dt} = -\frac{1}{\alpha}\frac{M}{\beta_3 M^{7/2}}; \Rightarrow \int_{M_i}^{0} M^{5/2}dM = -\beta_4\int_0^T dt; \Rightarrow -\frac{2}{7}M_i^{7/2} = -\beta_4(T) \Rightarrow T \propto M_i^{7/2}$$

So the lifetime of the cluster is $T \propto M_i^{7/2}$ ■.

Now the density $\rho \propto \frac{M}{R^3}$. But for a system in dynamical equilibrium we have $R \propto M^2$. This gives $\rho \propto \frac{M}{(M^2)^3} = M^5 \Rightarrow M \propto \rho^{-5}$ Eq. (2.13) can be solved as a function of time as above and written as

$$M = \beta_5 t^{2/7} \Rightarrow M^{-5} = \beta_5 t^{-10/7} \Rightarrow \rho = \beta_6 t^{-10/7}$$

□

(c) Estimate this for a globular cluster of mass $5 \times 10^5 M_\odot$ radius $10pc$ and mean stellar mass $0.5M_\odot$
Solution:
The density of this cluster is

$$\rho \approx \frac{M}{R^3} = \frac{5 \times 10^5 M_\odot}{10^3 pc^3} = 9.86 \times 10^{-14} kg/m^3 = 5 \times 10^2 M_\odot/pc^3$$

. The number of star is

$$N = \frac{M_{tot}}{m_{av}} = \frac{5 \times 10^5 M_\odot}{0.5 M_\odot} = 1 \times 10^6$$

The time scale then is

$$t = \left(\frac{GM}{R^3}\right)^{-1/2} = \frac{6.67 \times 10^{-11} \cdot 5 \times 10^5 M_\odot}{10^3 pc^3} = 6.67 \times 10^5 yr$$

□

2.5 Homework Five

2.5.1. (a) Calculate the total gravitational potential energies of (i) a homogeneous sphere of mass M and radius a, and (ii) a Plummer sphere of mass M and scale length a

Solution:

The potential energy is $U = \frac{GM(r)m}{r}$. Where $M(r)$ is the mass inside of spherical shell of radius r. For a homogenous spherical distribution of ρ the $M(r) = \frac{4}{3}\pi r^3 \rho$ and the additional mass increase due to increase in the radius of mass is $dm = \rho 4\pi r^2 dr$.If we bring dm from infinity to r then the increase in potential energy is

$$dU = \frac{GM(r)}{r} dm = \frac{GM(r)}{r} \cdot \rho 4\pi r^2 dr = \frac{G\frac{4}{3}\pi r^3 \rho}{r} \cdot \rho 4\pi r^2 dr \tag{2.15}$$

The total potential energy is obtained by integrating Eq. (2.15) from 0 to the radius of the final sphere a.

$$U = \int_0^a \frac{16}{3}\pi^2 \rho^2 G r^4 dr = \frac{16}{3}\pi^2 G \rho^2 \frac{a^5}{5} = \frac{16\pi^2 G a^5}{15}\rho^2 \tag{2.16}$$

But for a homogenous sphere of radius a the density is $\rho = \frac{3M}{4\pi a^3}$. Using this is Eq. (2.16) we get

$$U = \frac{16}{3}\pi^2 G \frac{a^5}{5}\left(\frac{3M}{4\pi a^3}\right)^2 = \frac{3GM^2}{5a}$$

So the gravitational potential energy of homogenous shpere of mass M and radius a is $\frac{3GM^2}{5a}$. ■

Given any potential function we can always calculate the density function using the poisson equation.

$$\Phi = \frac{GM}{\sqrt{r^2 + a^2}} \quad \text{Plummer Potential} \tag{2.17}$$

$$\nabla^2 \Phi = 4\pi G \rho(r) \quad \text{Poisson's equation} \tag{2.18}$$

For spherical system the Laplacian operator is $\nabla^2 := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$. Calculating $\frac{\partial \phi}{\partial r}$ we have.

$$\frac{\partial \phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{GM}{\sqrt{r^2 + a^2}}\right) = -\frac{GMr}{(r^2 + a^2)^{3/2}}; \Rightarrow r^2 \frac{\partial \phi}{\partial r} = -\frac{GMr^3}{(r^2 + a^2)^{3/2}}$$

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{GMr^3}{(r^2 + a^2)^{3/2}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[GM\left(1 + \frac{a^2}{r^2}\right)^{-\frac{3}{2}}\right] = \frac{3GMa^2}{(r^2 + a^2)^{5/2}}$$

Poisson's equation can be used to calculate the density function as $\rho(r) = \frac{\nabla^2 \Phi}{4\pi G}$.

$$\rho(r) = \frac{1}{4\pi G} \cdot \frac{3GMa^2}{(r^2 + a^2)^{5/2}} = \frac{3Ma^2}{4\pi}\frac{1}{(r^2 + a^2)^{5/2}} \tag{2.19}$$

Eq.(2.19) gives the density function of the plummer model. This density function can be used to calculate the mass of spherical volume of radius r as:

$$M(r) = \int_0^r \rho(r) 4\pi r^2 dr = 4\pi \int_0^r \frac{3Ma^2}{4\pi} \frac{r^2}{\left(r^2+a^2\right)^{5/2}} dr = \frac{Mr^3}{\left(r^2+a^2\right)^{3/2}} \tag{2.20}$$

We can use Eq.(2.15) to calculate the potential energy equipped with the mass function and density function.

$$U = 4\pi G \int_0^\infty \frac{Mr^3}{\left(r^2+a^2\right)^{3/2}} \cdot r^2 \cdot \frac{3Ma^2}{4\pi} \frac{1}{\left(r^2+a^2\right)^{5/2}} dr = 3GM^2a^2 \int_0^\infty \frac{r^4}{\left(r^2+a^2\right)^4} = \frac{3\pi}{32}\frac{GM^2}{a}$$

So the total gravitational energy of plummer potential function is $\frac{3\pi}{32}\frac{GM^2}{a}$. □

(b) Show that the total mass of the Plummer model is indeed M.
Solution:
Eq.(2.20) gives the mass cantained within the radius r for plummer sphere. The total mass of plummer sphere is the total mass contained inside the radius of $r = \infty$. Taking limit of Eq.(2.20) we get.

$$M_{tot} = \lim_{r\to\infty} \frac{Mr^3}{\left(r^2+a^2\right)^{3/2}} = \lim_{r\to\infty} \frac{M}{\left(1+\frac{a^2}{r^2}\right)^{\frac{3}{2}}} = M$$

This shows that the total mass of plummer model is M which appears in the potential function given by Eq. (2.17). □

2.5.2. (a) Verify that the Kuzmin potential

$$\Phi_K(r,z) = -\frac{GM}{\sqrt{r^2+(a+|z|)^2}} \tag{2.21}$$

has $\nabla^2\Phi = 0$ for $z \neq 0$, and so represents a surface density distribution $\Sigma(r)$ in the plane $z = 0$.
Solution:
Writing $r^2 = x^2+y^2$ where x and y are the cartesian coordinates corresponding to the r coordinate in cylindrical system. We get $\Phi = -GM(x^2+y^2+(a+|z|)^2)^{-1/2}$. In cartesian coordinate system $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. So each components of ths operator are.

$$\frac{\partial^2}{\partial x^2}\Phi = \frac{GM\left(2x^2-y^2-(a+z)^2\right)}{\left(x^2+y^2+(a+z)^2\right)^{\frac{5}{2}}}; \qquad \frac{\partial^2}{\partial y^2}\Phi = \frac{GM\left(-x^2+2y^2-(a+z)^2\right)}{\left(x^2+y^2+(a+z)^2\right)^{\frac{5}{2}}}$$

Since the potential is function of $|z|$ and the derivative of $|z|$ dosent't exist at $z = 0$. We take left hand and right hand derivative for the z component. Using $|z| = +z$ for right and $|z| = -z$ for left derivative, We get.

$$\frac{\partial^2}{\partial z_+^2}\Phi = \frac{GM\left(-x^2-y^2+2\,(a+z)^2\right)}{\left(x^2+y^2+(a+z)^2\right)^{\frac{5}{2}}} \quad \frac{\partial^2}{\partial z_-^2}\Phi = \frac{GM\left(-x^2-y^2+2\,(a-z)^2\right)}{\left(x^2+y^2+(a-z)^2\right)^{\frac{5}{2}}}$$

In each of the cases the total sum

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_+^2}\right)\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_-^2}\right)\Phi = 0$$

By use of Poisson's equation $\rho(r) = 1/4\pi G\nabla^2\Phi$ we conclude the mass density is zero everywhere except (possibly?) at $z = 0$. □

(b) Use Gauss's law to determine $\Sigma(r)$.
Solution:
The gauss law for gravitational field says $\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = 4\pi G M_{encl}$ where S is any arbitrary closed surface and M_{encl} is the mass inside that surface. Now that we know that there is no mass except at infinite plane $z = 0$, we are certain that the Gravitational force field is completely along $\hat{z}$. The force field along $\hat{z}$ is given by $\boldsymbol{E} = \frac{\partial \Phi}{\partial z}$. Since the potential function is not smooth, we have two different values for this derivative on either side of the disc.

$$\boldsymbol{E}_+ = \frac{\partial \Phi}{\partial z_+}(-\hat{z}) = \frac{\partial}{\partial z_+}\left(\frac{GM}{\sqrt{r^2 + (a+z)^2}}\right)(-\hat{z}) = -\frac{GM\,(a+z)}{\left(r^2 + (a+z)^2\right)^{3/2}}(-\hat{z})$$

$$\boldsymbol{E}_- = \frac{\partial \Phi}{\partial z_-}\hat{z} = \frac{\partial}{\partial z_-}\left(\frac{GM}{\sqrt{r^2 + (a-z)^2}}\right)\hat{z} = \frac{GM\,(a-z)}{\left(r^2 + (a-z)^2\right)^{3/2}}$$

If we take a cylindrical gaussian surface for S with surface Area $A\hat{z}$, The total mass inside the cylinder is $M_{encl} = \Sigma \times A$ and the flux though the surface $\oint \boldsymbol{E} \cdot d\boldsymbol{A} = E_+ A + E_- A$. But E_z is uniform so we can calculate $E_+ = \frac{\partial \Phi}{\partial z^+}\Big|_{z=0} = \frac{GMa}{(r^2+a^2)^{3/2}}$. And similarly for $E_- = \frac{GMa}{(r^2+a^2)^{3/2}}$ So,

$$4\pi G \Sigma \times A = \frac{GMa}{(r^2+a^2)^{3/2}}A + \frac{GMa}{(r^2+a^2)^{3/2}}A; \Rightarrow \Sigma = \frac{Ma}{2\pi\,(r^2+a^2)^{3/2}}$$

So the surface mass density of the Kuzmin disk is $\Sigma(r) = \frac{Ma}{2\pi(r^2+a^2)^{3/2}}$. □

(c) What is the circular orbit speed for a particle moving in the plane of the disk?
Solution:
For this potential the total mass inside the spherical shell of radius r is simply the surface density times the area of great circle, so $M(r) = \Sigma(r)\pi r^2$. The transverse speed for a circular orbit

$$v_c = \sqrt{\frac{GM(r)}{r}} = \sqrt{\frac{G\pi r^2}{r} \cdot \frac{Ma}{2\pi\,(r^2+a^2)^{3/2}}} = \sqrt{\frac{GMar}{2\,(r^2+a^2)^{3/2}}}$$

This gives the speed of particle in circular orbit for Kuzmin potential. □

2.5.3. For stars moving vertically in Galactic disk, with energy $E_z = \Phi(R_0, z) + {}^1\!/_2 v_z^2$, suppose the distribution function is

$$f(z, v_z) = \frac{n_0}{\sqrt{2\pi\sigma^2}} e^{-E_z/\sigma^2}.$$

Find the density $n(z)$ and give it's value $n(0)$. To construct self consistent model let $\Phi(z) = \sigma^2\phi$, show that

$$2\frac{d^2\phi}{dy^2} = e^{-\phi}, \qquad \text{Where } y = \frac{z}{z_0} \qquad \text{and} \quad z_0^2 = \frac{\sigma^2}{8\pi G m n_0}$$

Solve this for $\phi(y)$ and hence find $\Phi(z)$ and $n(z)$. What is the value at large $|z|$?
Solution:
The number density is the zeroth moment of this distribution function so

$$n(z) = \int_{-\infty}^{\infty} f(z, v_z)dv_z = \frac{n_0}{\sqrt{2\pi\sigma^2}}\, 2\int_0^{\infty} e^{(-\Phi - {}^1\!/_2 v_z^2)/\sigma^2} = \frac{2n_0 e^{-\Phi/\sigma^2}}{\sqrt{2\pi\sigma^2}} \underbrace{\int_0^{\infty} e^{-\frac{v_z^2}{2\sigma^2}}}_{\sigma\sqrt{\frac{\pi}{2}}} = n_0 e^{-\Phi(R,z)/\sigma^2}$$

This gives the expression for $n(z)$. Since $\Phi(z=0) = 0$ is given. $n(0) = n_0 e^{-\Phi(z=0)/\sigma^2} = n_0 e^0 = n_0$.

The total mass density $4\pi G\rho(z) = \nabla^2\Phi$. But $\rho(z) = mn(z)$ where m is the average mass. But for motion only along z, we can write $\nabla^2\Phi \equiv \frac{d^2\Phi}{dz^2}$. Also operator $dz^2 = z_0^2 dy^2$;. By poissons equation,

$$\frac{d^2\Phi}{dz^2} = 4\pi Gmn(z); \quad \sigma^2\frac{d^2\phi}{z_0^2dy^2} = 4\pi Gmn_0e^{-\phi}; \quad \Rightarrow 2\frac{d^2\phi}{dy^2} = e^{-\phi(y)} \text{ if } z_0^2 = \frac{\sigma^2}{8\pi Gmn_0}$$

Now solving this for ϕ as a function of y

$$2\frac{d^2\phi}{dy^2} = e^{-\phi(y)}$$

This differential equation should give a function $\phi(y)$ such that $n(z) = n_0e^{-\phi(z_0y)/\sigma^2} = n_0 sech^2(z/(2z_0))$ but I couldn't find any reasonable solution

For large value of $|z|$

$$\lim_{z_+\to\infty} n_0 sech^2\left(\frac{z}{2z_0}\right) = 0; \qquad \lim_{z_-\to\infty} n_0 sech^2\left(-\frac{z}{2z_0}\right) = 0$$

So for large value of $|z|$ the density is zero. □

2.5.4. A stellar system in which all particles are on radial orbits is described by the distribution function

$$f(\mathcal{E}, L) = \begin{cases} A\delta(L)(\mathcal{E} - \mathcal{E}_0)^{-1/2} & \text{if } \mathcal{E} > \mathcal{E}_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{E} = \psi - 1/2v_t^2$is relative energy and $\mathcal{E}_0$ and A are constants.

(a) By writing $v^2 = v_r^2 + v_t^2$, where v_r and v_t are the radial and transverse velocities, and $L = rv_t$, prove that the volume element $d^3v = 2\pi v_t dv_t dv_r$ may be written $d^3v = \frac{\pi d\mathcal{E}dX}{r^2v_r}$ where $X = L^2$.

(b) Hence show that the density is

$$\rho(r) = \begin{cases} Br^{-2} & \text{if } (r < r_0) \\ 0 & \text{if } (r \geq r_0) \end{cases}$$

where B is a constant and the relative potential at r_0 satisfies $\psi(r_0) = \mathcal{E}_0$.

Solution:

The number density is the zeroth moment of distribution function with respect to velocity. So

$$n(z) = \int f d^3v = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\delta(L)(\mathcal{E} - \mathcal{E}_0)^{-1/2}d^3v = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\delta(\sqrt{X})(\mathcal{E} - \mathcal{E}_0)^{-1/2}\frac{\pi d\mathcal{E}dX}{r^2v_r}$$

$$= \int_{-\infty}^{\infty} A\left(\int_{-\infty}^{\infty}\delta(\sqrt{X})dX\right)(\mathcal{E} - \mathcal{E}_0)^{-1/2}\frac{d\mathcal{E}}{r^2v_r}$$

If $r_0 < r$ then $\mathcal{E} = \psi(r) > \mathcal{E}_0$

$$\rho(r) = mn(r) = \int_{\mathcal{E}_0}^{\infty} \frac{mA(\mathcal{E} - \mathcal{E}_0)^{-1/2}}{r^2v_r}d\mathcal{E}$$

$$= \frac{1}{r^2}\left[\frac{-2m(\mathcal{E} - \mathcal{E}_0)^{-3/2}}{3v_r}\right]_{\mathcal{E}_0}^{\infty} = Br^{-2}$$

But if $r_0 < r$ then $\mathcal{E} = \psi(r) < \mathcal{E}_0$ and $f(\mathcal{E}, L) = 0$ then,

$$\rho(r) = \int 0 d^3 v = 0$$

This is a power law density with density decaying as square of the distance for a finite spherical region in space. □

2.6 Homework Six

2.6.1. Assuming the rotation curve for milky way is flat and$V(R) = R\Omega(R) = 200km/s$ and $R_0 = 8kpc$. (a) Compute the Oort constants A and B, and the local epicyclic frequency κ. (b) If the Sun has v_x (radial) $= 10km/s$ and v_y (transverse) $= 5km/s$, calculate the Sun's guiding radius R g and radial orbital amplitude X.

Solution:

For flat rotation curve $v(r) =$ constant. so, $\frac{dv}{dr} = 0$. .

$$A = \frac{1}{2}\frac{V(R)}{R_0} = \frac{1}{2}\frac{200km/s}{8kpc} = 12.50km/s/kpc$$
$$B = -\frac{1}{2}\frac{V(R)}{R_0} = -\frac{1}{2}\frac{200km/s}{8kpc} = -12.50km/s/kpc$$

The value of κ is related to the oort constant as $\kappa^2 = -4B\Omega$

$$\Omega = V(R)/R = 200/8 = 2.5km/s/kpc; \qquad \kappa = \sqrt{4 * 12.50 * 2.5} = 11.18km/s/kpc$$

Also

$$v_y = 2BX; \Rightarrow X = \frac{5km/s}{2 \cdot 12.50km/s/kpc} = 0.2kpc$$

The guiding center is the sum of maximum displacement X and the closest approach so $R_g = R_0 + X = 8kpc + 0.2kpc = 8.20kpc$. □

2.6.2. Show that, if the rotation curve of the Milky Way is flat near the Sun, then $\kappa = \sqrt{2}\Omega(R)$, so that locally $\kappa \approx 36km/s/kpc$. Sketch the curves ofΩ, $\Omega \pm \kappa/2$, and$\Omega \pm \kappa/4$ in a disk where $V(R)$ is constant everywhere, and show that the zone where two-armed spiral waves can persist is almost four times larger than that for four-armed spirals.

Solution:

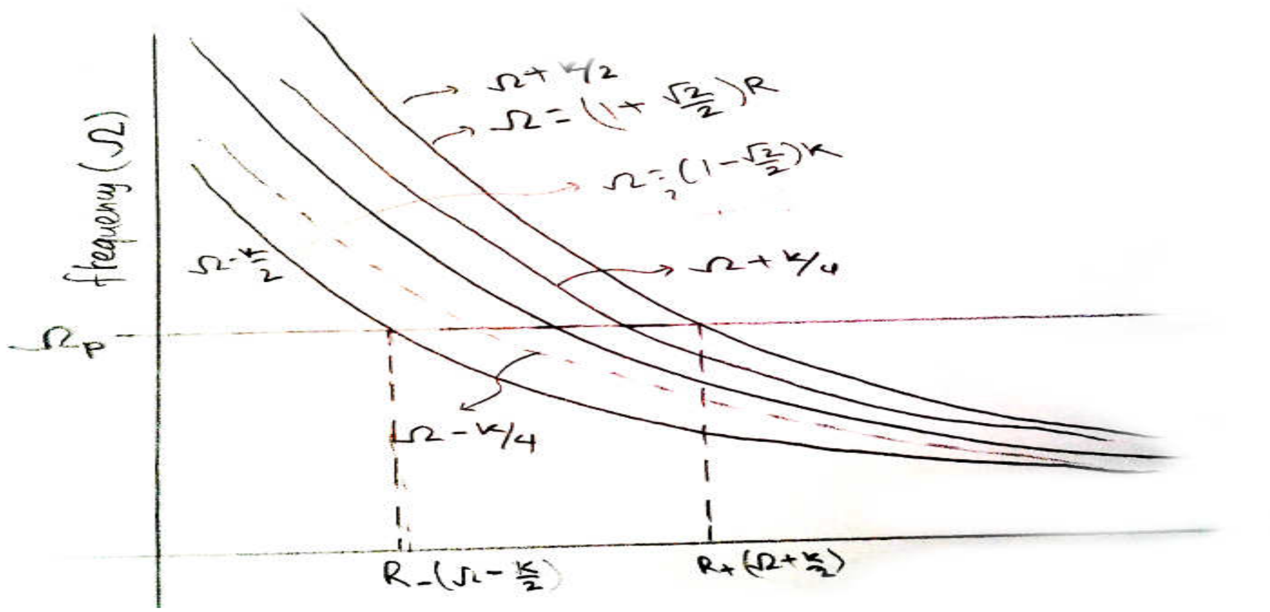
For a flat rotation curve $V(R)$ =constant, so, $\frac{dV(R)}{dR} = 0$. The oort constant B is $B = -\frac{1}{2}\frac{V(R)}{R} = -\frac{\Omega}{2}$. But $\kappa^2 = -4B\Omega$. This gives

$$\kappa = \sqrt{-4 \cdot -\frac{\Omega}{2} \cdot \Omega} = \sqrt{2}\Omega(R)$$

This gives the epicyclic frequency of the sun. The graph for $\Omega \pm \frac{\kappa}{2}$ and $\Omega \pm \frac{\kappa}{4}$ are The lowest and highest values of R can be found at the points where Ω crosses the pattern speed Ω_p. The point $\Omega \pm \frac{\kappa}{2}$ crosses Ω_p are $R_{max} = (1 \pm \frac{1}{\sqrt{2}})R$ This gives the ration of region as

$$\frac{R_{max}}{R_{min}} = \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = 5.8$$

Similarly The point $\Omega \pm \frac{\kappa}{4}$ crosses Ω_p are $R_{max} = (1 \pm \frac{1}{2\sqrt{2}})R$ This gives the ration of region as



$$\frac{R_{max}}{R_{min}} = \frac{1 + \frac{1}{2\sqrt{2}}}{1 - \frac{1}{2\sqrt{2}}} = 2.09$$

The region are approximately at the ratio of 3.0 □

2.6.3. (a) Given the dispersion relation for a gas disk, $(\omega - m\Omega)^2 = k^2 v_s^2 - 2\pi G\Sigma|k| + \kappa^2$, Show that the group velocity is

$$v_g \equiv \left.\frac{\partial \omega}{\partial k}\right|_R = \text{sign}(k)\frac{|k|v_s^2 - \pi G\Sigma}{\omega - m\Omega}.$$

Solution:
Differentiating both sides of the given dispersion relation with respect to k, gives

$$2(\omega - m\Omega)\frac{\partial \omega}{\partial k} = 2kv_s^2 - 2\pi G\Sigma \text{sign}(k)$$

For any real number k we can write $k = |k|\text{sign}(k)$ using this in above expression can be rearranged in the form

$$\frac{\partial \omega}{\partial k} = \frac{2|k|\text{sign}(k)v_s^2 - 2\pi G\Sigma \ \text{sign}(k)}{2(\omega - m\Omega)} = \text{sign}(k)\frac{|k|v_s^2 - 2\pi G\Sigma}{\omega - m\Omega}$$

This gives the required group velocity as required. □

(b) Show that, for a mirginally stable disk with $Q = \dfrac{v_s \kappa}{\pi G\Sigma} = 1$ the group velocity is equal to the sound speed v_s

Solution:
For $Q = 1$ we have $\pi G\Sigma = v_s\kappa$. Using this in the expression of group velocity gives

$$v_g = \text{sign}(k)\frac{|k|v_s^2 - v_s\kappa}{\omega - m\Omega}$$

We can use $\kappa = m\Omega$ and $k = \frac{\omega}{v_s}$. If we disregard the sign of k (ie, assume k as positive) the above expression becomes

$$v_g = \text{sign}(k)\frac{\frac{\omega}{v_s}v_s - m\Omega}{\omega - m\Omega}v_s = \frac{\omega - m\Omega}{\omega - m\Omega}v_s = v_s$$

This shows that the group velocity is (within a signof k) equal to the sound speed. □

2.6.4. A satellite galaxy of mass M_s moves in a circular orbit of radius R in a spherically symmetric galactic halo of density $\rho(r) = v_c^2/4\pi G r^2$, with $M_s \ll v_c^2 R/G$. The stars (and dark matter particles) in the parent galaxy all have masses much less than M_s .

(a) Use the equation for dynamical friction to write down the drag force on the satellite as it orbits.
Solution:
The dynamical friction is given by,

$$-\frac{dv}{dt} = \frac{4\pi G(M_s + m)}{v^2} nm \ln(\Lambda);$$

For a satellite galaxy of mass M_s obiting at v_c the passing velocity is $V = v_c$ the drag force is $-M_s\frac{dv_c}{dt}$. Noting that for the galactic halo $nm = \rho(r)$ leads to.

$$F_{drg} = -M_s\frac{4\pi G^2 \cancelto{M_s}{(M_s + m)}}{v_c^2} \cdot \frac{v_c^2}{4\pi G r^2}\ln(\Lambda) = -\frac{M_s^2 G}{r^2}\ln(\Lambda)$$

This gives the expression for the drag force on the orbiting galaxy in the halo. □

(b) The satellite sinks inward so slowly that it can be thought of as moving through a series of circular orbits, so its orbital speed at any radius r is always equal to the circular orbital speed at r. What is the angular momentum $L(r)$ of the satellite at radius r?
Solution:
The instantaneous speed at a distance r from the center is v_c, so the momentum is $P = M_s v_c$. The angular momentum is $L = r \times p$

$$L = r \times P = r M_s v_c \tag{2.22}$$

So the angular momentum of the galaxy at distance r is $M_s v_c r$ □

(c) By equating the rate of change of L to the torque exerted on the satellite by dynamical friction, show that the distance $r(t)$ from the satellite to the center of the galaxy obeys the differential equation

$$\frac{dr}{dt} = -\frac{GM_s \ln(\Lambda)}{v_c r}$$

Solution:
The torque about the center of the galactic halo which the galaxy is orbiting is $\tau = F_{drg} r$, but $\tau = \frac{dL}{dt}$, combining these two give

$$\frac{dL}{dt} = F_{drg} r; \Rightarrow M_s v_c \frac{dr}{dt} = -\frac{M_s^2 G}{r^2}\ln(\Lambda) \cdot r; \Rightarrow \frac{dr}{dt} = -\frac{GM_s \ln(\Lambda)}{v_c r}$$

Which the required differential equation for the rate of change of distance of orbiting galaxy to center of halo. □

(d) Solve this equation to estimate the time taken for the satellite to sink to the center of the parent galaxy.
Solution:
The time to fall t_f into the center of halo is given by the time for the distance of R_0 to 0 at the center of halo. Rearranging the above differential equation we get.

$$rdr = -\frac{GM_s \ln(\Lambda)}{v_c} dt; \Rightarrow \int_{R_0}^{0} rdr = -\int_{0}^{t_f} \frac{GM_s \ln(\Lambda)}{v_c} dt; \Rightarrow -\frac{{R_0}^2}{2} = \frac{GM_s \ln(\Lambda)}{v_c} t_f$$

So the time to sink is $t_f = \frac{R_0^2 v_c}{2GM_s \ln(\Lambda)}$. □

(e) Evaluate this time for a hypothetical "Magellanic Cloud" with $M_s = 2 \times 10^{10} M_\odot$ on an initially circular orbit of radius $R = 50kpc$ around our Galaxy, with $v_c = 220km/s$. Take $\Lambda = 20$.
Solution:
Substuting these values in the above expression

$$t_f = \frac{(50 \times 10^3)^2 \cdot 220 \times 10^3}{2 \cdot 6.67 \times 10^{-11} \cdot 2 \times 10^{10} M_\odot} = 3.28 \times 10^{16} s = 1.04 \times 10^9 yr = 1.04 Gyr$$

So the sink time of the cloud is $1.04Gyr$ □

2.6.5. If the effective radius of the satellite galaxy in the previous problem is $R_s = 1.5kpc$, estimate the distance from the center of the parent galaxy at which tidal (differential) gravitational forces would significantly affect the satellite's structure.
Solution:
The distance scale is given by

$$r_t = \left(\frac{M}{M_s}\right)^{\frac{1}{3}} R_s$$

Assuming $M_s = 2 \times 10^{10} M_\odot$ froom previous problem and the mass of galaxy to be that of Milky way $M = 5.8 \times 10^{11} M_\odot$

$$r_t = \left(\frac{5.8 \times 10^{11} M_\odot}{2 \times 10^{10} M_\odot}\right)^{\frac{1}{3}} 1500pc = 488.2pc$$

So the distance for significant effect is $488.2pc$ □

Chapter 3

Quantum Mechanics

3.1 Homework One

3.1.1. (a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle, \langle a''|\alpha\rangle, \cdots$ and $\langle a'|\beta\rangle, \langle a''|\beta\rangle, \cdots$ are all known, where $|a'\rangle, |a''\rangle, \cdots$ form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in this basis.

Solution:

We know every ket can be written as the sum of its component in the 'direction' of base ket (completeness) so $|\gamma\rangle$ can be written as

$$|\gamma\rangle = \sum_i |a^i\rangle \langle a^i|\gamma\rangle$$

Let the operator $|\alpha\rangle\langle\beta|$ act on an arbitrary ket $|\gamma\rangle$.

$$|\alpha\rangle\langle\beta|\gamma\rangle = \sum_i |\alpha\rangle \langle\beta|a^i\rangle \langle a^i|\gamma\rangle$$

So the component of this $|\alpha\rangle\langle\beta|\gamma\rangle$ in the direction of another eigen ket $|a^j\rangle$ is then given by the inner product of it with $|a^j\rangle$

$$(|\alpha\rangle\langle\beta|\gamma\rangle)_j = \langle a^j|\alpha\rangle\langle\beta|\gamma\rangle = \sum_i \underbrace{\langle a^j|\alpha\rangle\langle\beta|a^i\rangle}_{N\times N}\langle a^i|\gamma\rangle \tag{3.1}$$

This anove expression can be written as the matrix form as

$$\begin{bmatrix} (|\alpha\rangle\langle\beta|\gamma\rangle)_1 \\ (|\alpha\rangle\langle\beta|\gamma\rangle)_2 \\ \vdots \\ (|\alpha\rangle\langle\beta|\gamma\rangle)_N \end{bmatrix} = \begin{bmatrix} \langle a^1|\alpha\rangle\langle\beta|a^1\rangle & \langle a^1|\alpha\rangle\langle\beta|a^2\rangle & \cdots & \langle a^1|\alpha\rangle\langle\beta|a^N\rangle \\ \langle a^2|\alpha\rangle\langle\beta|a^1\rangle & \langle a^2|\alpha\rangle\langle\beta|a^2\rangle & \cdots & \langle a^2|\alpha\rangle\langle\beta|a^N\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a^N|\alpha\rangle\langle\beta|a^1\rangle & \langle a^N|\alpha\rangle\langle\beta|a^2\rangle & \cdots & \langle a^N|\alpha\rangle\langle\beta|a^N\rangle \end{bmatrix} \begin{bmatrix} (|\gamma\rangle)_1 \\ (|\gamma\rangle)_2 \\ \vdots \\ (|\gamma\rangle)_N \end{bmatrix}$$

Since every $\langle a'|\beta\rangle$ is known each element $\langle\beta|a^i\rangle$ in above matrix can be written as the complex conjugate of known $\langle a^i|\beta\rangle^*$. So the matrix representation becomes

$$|\alpha\rangle\langle\beta| \equiv \begin{bmatrix} \langle a^1|\alpha\rangle\langle a^1|\beta\rangle^* & \langle a^1|\alpha\rangle\langle a^2|\beta\rangle^* & \cdots & \langle a^1|\alpha\rangle\langle a^N|\beta\rangle^* \\ \langle a^2|\alpha\rangle\langle a^1|\beta\rangle^* & \langle a^2|\alpha\rangle\langle a^2|\beta\rangle^* & \cdots & \langle a^2|\alpha\rangle\langle a^N|\beta\rangle^* \\ \vdots & \vdots & \ddots & \vdots \\ \langle a^N|\alpha\rangle\langle a^1|\beta\rangle^* & \langle a^N|\alpha\rangle\langle a^2|\beta\rangle^* & \cdots & \langle a^N|\alpha\rangle\langle a^N|\beta\rangle^* \end{bmatrix}$$

Which is the required matrix representation of $|\alpha\rangle\langle\beta|$ $\square$

(b) Consider of spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|S_z = \hbar/2\rangle$ and $|S_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (S_z diagonal) basis.
Solution:
The basis kets are $|S_z;+\rangle \equiv |+\rangle$ and $|S_z;-\rangle \equiv |-\rangle$. The state ket $|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$. So the four matrix elements are

$$\langle\alpha|+\rangle = 1; \qquad \langle\alpha|-\rangle = 0$$
$$\langle\beta|+\rangle = \frac{1}{\sqrt{2}}(1+0) = \frac{1}{\sqrt{2}} \qquad \langle\beta|-\rangle = \frac{1}{\sqrt{2}}(0+1) = \frac{1}{\sqrt{2}}$$

The required matrix representation is

$$\begin{bmatrix} \langle+|\alpha\rangle\langle\beta|+\rangle & \langle+|\alpha\rangle\langle\beta|-\rangle \\ \langle-|\alpha\rangle\langle\beta|+\rangle & \langle-|\alpha\rangle\langle\beta|-\rangle \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Which is the required matrix representation of the operator in the basis $|S_z;+\rangle$ and $|S_z;-\rangle$ □

3.1.2. Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k, \qquad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij},$$

Where, $S_x = \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|), \quad S_y = \frac{i\hbar}{2}(-|+\rangle\langle-| + |-\rangle\langle+|), \quad S_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$

Solution:

$$S_xS_y = \frac{i\hbar^2}{4}\{-|+\rangle\langle-|+\rangle\langle-| + |+\rangle\langle-|-\rangle\langle+| - |-\rangle\langle+|+\rangle\langle-| + |-\rangle\langle+|-\rangle\langle+|\} = \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\}$$
$$S_yS_x = \frac{i\hbar^2}{4}\{|+\rangle\langle-|+\rangle\langle-| - |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|+\rangle\langle-| - |-\rangle\langle+|-\rangle\langle+|\} = -\frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\}$$
$$[S_x, S_y] = S_xSy - S_yS_x = \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} + \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} = \frac{i\hbar^2}{2}\{|+\rangle\langle-| - |-\rangle\langle-|\} = i\hbar S_z$$

Since $[S_x, S_y] = i\hbar S_z$ it immediately follows that $[S_y, S_x] = -i\hbar S_z$ because $[A, B] = -[A, B]$. Collecting all these leads to $[S_i, S_j] = i\varepsilon_{ijk}S_k$.

$$\{S_x, S_y\} = S_xS_y + S_yS_x = \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} - \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} = 0$$
$$\{S_x, S_x\} = S_xS_x + S_xS_x = 2S_xS_x = 2\frac{\hbar^2}{4}\left\{\underbrace{\langle+|+\rangle + \langle-|-\rangle}_{\text{Identity operator}}\right\} = \frac{\hbar^2}{2}$$

Similarly $\{S_x, S_x\} = \frac{\hbar^2}{2}; \{S_y, S_y\} = \frac{\hbar^2}{2}; \{S_z, S_z\} = \frac{\hbar^2}{2}; \{S_x, S_y\} = 0; \{S_y, S_z\} = 0; \{S_z, S_x\} = 0$; which can be copactly written as $\{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij}$ for each operator leads to the required relation of the commutation and anti commutation relation of the given operators. □

3.1.3. The hamiltonian operator for a two-state system is given by

$$h = a(|1\rangle\langle1| - |2\rangle\langle2| + |1\rangle\langle2| + |2\rangle\langle1|),$$

where a is a number with the dimension of energy. find the energy eigenvalues and the corresponding energy eigenkets (as a linear combinations of $|1\rangle$ and $|2\rangle$)

Solution:
let the energy eigenket be $|\alpha\rangle = p\,|1\rangle + q\,|2\rangle$. let the eigenvalue of this energy eigenket be a'. operating this eigenket by the given hamiltonian operator we get.

$$\begin{aligned} h\,|\alpha\rangle &= a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)(p\,|1\rangle + q\,|2\rangle) \\ &= a(p\,|1\rangle + p\,|2\rangle - q\,|2\rangle + q\,|1\rangle) \\ &= a[(p+q)\,|1\rangle + (p-q)\,|2\rangle] \end{aligned}$$

since we assume a' is the eigenvaue of this ket we must have $h\,|\alpha\rangle = a'\,|\alpha\rangle$ thus

$$a'(p\,|1\rangle + q\,|2\rangle) = a[(p+q)\,|1\rangle + (p-q)\,|2\rangle]$$

since $|1\rangle$ and $|2\rangle$ are independent kets, the coefficient of each ket on lhs and rhs must equal. comparing the coefficients we have

$$\begin{aligned} -ap + (a+a')q = 0; &\qquad \to p = \frac{a+a'}{a}q \\ (a-a')p + aq = 0; &\qquad \to (a-a')\frac{a+a'}{a}q + aq = 0; \qquad \to a^2 - a'^2 + a^2 = 0; \quad a' = \pm\sqrt{2}a \end{aligned}$$

so the required eigenvalues of the operator are $\pm\sqrt{2}a$.

the coefficient

$$p = \frac{a \pm \sqrt{2}a}{a}q = (1 \pm \sqrt{2})q$$

. since we have a free choice of one of the parameters we choose p and q such that the energy eigenket is normalized. so the required eigenket is

$$|\alpha\rangle = \frac{1}{\sqrt{1+(1\pm\sqrt{2})^2}}((1\pm\sqrt{2})\,|1\rangle + |2\rangle) = \frac{1}{\sqrt{4\pm 2\sqrt{2}}}\left((1\pm\sqrt{2})\,|1\rangle + |2\rangle\right)$$

the above expression $|\alpha\rangle$ gives the energy eigenket corresponding to eigenvalue $\pm\sqrt{2}a$. □

3.1.4. A beam of spin $\frac{1}{2}$ atom goes through a series of stern-gerlach-type measurements as follows:

(a) the first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.
(b) the second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\boldsymbol{s}\cdot\hat{\boldsymbol{n}}$ with $\hat{\boldsymbol{n}}$ making an angle β in the $xz-$plane wi respect to the z-axis.
(c) the third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

what is the intensity of the final $s_z = -\hbar/2$ bean when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? how must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?
Solution:
The First Stern-Gerlach measurement in S_z is independent of the second Stern-Gerlach measurement in $\hat{\boldsymbol{n}}$ the probability of atom passing through each component is $\frac{1}{2}$. Due to this measurement and the $S_n = -\hbar/2$ being rejected the system essentially forgets the previous measurement and the atom still come out 50%. So the fraction of atoms passing through the third SG apparatus in S_z direction is still $\frac{1}{2}$. So the total fraction of atoms passing thourhg the third SG apparatus is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = 25\%$,

If the second SG apparatus is oriented parallel to the first apparatus then it essentially measures the $|S_z;+\rangle$ state of the atom which was what came from the first apparatus so it lets 100% of the atom in $|S_z;+\rangle$ state. And the third apparatus will let half of the second which is 50% of the atoms which passed through the first apparatus. Orienting the second SG apparatus parallel to the first will let all of the atoms, this is the required condition of maximizing the output of third. □

3.1.5. Prove that if operator $X = |\beta\rangle\langle\alpha|$, then the hermitian conjugate of the operator is $X^\dagger = |\alpha\rangle\langle\beta|$.
Solution:
Acting this operator $X = |\beta\rangle\langle\alpha|$ on an arbitrary ket $|\gamma\rangle$

$$\begin{aligned}
& X|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle \\
\Rightarrow\quad & \langle\gamma|X^\dagger = \langle\beta|\langle\alpha|\gamma\rangle^* && (\because \text{Dual correspondence}) \\
\Rightarrow\quad & \langle\gamma|X^\dagger = \langle\beta|\langle\gamma|\alpha\rangle && (\because \langle\gamma|\alpha\rangle = \langle\alpha|\gamma\rangle^*) \\
\Rightarrow\quad & \langle\gamma|X^\dagger = \langle\gamma|\alpha\rangle\langle\beta| && (\because c|\delta\rangle = |\delta\rangle c) \\
\Rightarrow\quad & \langle\gamma|X^\dagger = \langle\gamma|(|\alpha\rangle\langle\beta|) && (\because \text{ Associative property}) \\
\Rightarrow\quad & X^\dagger = |\alpha\rangle\langle\beta|
\end{aligned}$$

Thus if $X = |\beta\rangle\langle\alpha|$ then $X^\dagger = |\alpha\rangle\langle\beta|$ is shown as required. □

3.2 Homework Two

3.2.1. A two state system is is characterized by a Hamiltonian $H_{11}|1\rangle\langle 1| + H_{12}(|1\rangle\langle 2| + |2\rangle\langle 1|) + H_{22}|2\rangle\langle 2|$ where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and the corresponding energy eigenvalues.
Solution:
Let the energy eigenket be $|E\rangle = p|1\rangle + q|2\rangle$ and the eigenvalues be λ. OPerating thsi state by the given Hamiltonian Operator we get

$$\begin{aligned}
H|E\rangle &= H_{11}|1\rangle\langle 1| + H_{12}(|1\rangle\langle 2| + |2\rangle\langle 1|) + H_{22}|2\rangle\langle 2|(p|1\rangle + q|2\rangle) \\
&= H_{11}p\langle 1\,|1\rangle|1\rangle + H_{11}q\langle 1\,|2\rangle|1\rangle + H_{12}p\langle 1\,|1\rangle|2\rangle + H_{12}p\langle 2\,|1\rangle|1\rangle + H_{12}q\langle 1\,|2\rangle|2\rangle \\
&\quad + H_{12}q\langle 2\,|2\rangle|1\rangle + H_{22}p\langle 2\,|1\rangle|2\rangle + H_{22}q\langle 2\,|2\rangle|2\rangle \\
&= H_{11}p|1\rangle + H_{12}p|2\rangle + H_{12}q|1\rangle + H_{22}q|2\rangle \\
&= (H_{11}p + H_{12}q)|1\rangle + (H_{12}p + H_{22}q)|2\rangle
\end{aligned}$$

Since by assumption λ is the eigenvalue of this state we have $H|E\rangle = \lambda|E\rangle$ which gives

$$\lambda p|1\rangle + \lambda q|2\rangle = (H_{11}p + H_{12}q)|1\rangle + (H_{12}p + H_{22}q)|2\rangle$$

Comparing the coefficient of each independent we get

$$\lambda p = (H_{11}p + H_{12}q); \qquad \lambda q = (H_{12}p + H_{22}q)$$

$$\begin{aligned}
\Rightarrow (\lambda - H_{11})p - H_{12}q = 0; \quad p = \frac{H_12}{\lambda - H_{11}}q \\
H_{12}p + (H_{22} - \lambda)q = 0; \quad \Rightarrow H_{12}\left(\frac{H_{12}}{\lambda - H_{11}}\right)q + (H_{22} - \lambda)q = 0;
\end{aligned}$$

Solving this for λ we get

$$\lambda = \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{H_{11}^2 - 2H_{11}H_{22} + 4H_{12}^2 + H_{22}^2}$$

These are the required eigenvalues of the given operator. This eigenvalues can be plugged back into the given equation to get the values of p and q.

$$q = 1; \qquad p = \frac{H_{12}}{\frac{H_{22} - H_{11}}{2} \pm \frac{1}{2}\sqrt{H_{11}^2 - 2H_{11}H_{22} + 4H_{12}^2 + H_{22}^2}}$$

So the required eigenstates are

$$|E\rangle = \left(\frac{H_{12}}{\frac{H_{22}-H_{11}}{2} \pm \frac{1}{2}\sqrt{H_{11}^2 - 2H_{11}H_{22} + 4H_{12}^2 + H_{22}^2}}\right)|1\rangle + |2\rangle$$

The above eigenstae can be normalized if required to get the Energy eigenket. □

3.2.2. (a) Compute $\left\langle(\Delta S_x)^2\right\rangle \equiv \left\langle S_x^2\right\rangle - \langle S_x\rangle^2$ where the expectation value is taken for the S_z+ state. Using your result check the generalized uncertainity relation

$$\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$$

with $A \to S_x, B \to S_y$.

Solution:

Let $|+\rangle$ represent the $|S_z;+\rangle$ state. Then the expectation value of S_x for $|S_z;+\rangle$ can be calculated as

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right); \qquad S_y = \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right); \qquad S_x = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right);$$

$$S_x|+\rangle = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle|+\rangle\right)|+\rangle = \frac{\hbar}{2}|-\rangle; \qquad S_x|-\rangle = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)|-\rangle = \frac{\hbar}{2}|+\rangle;$$

$$S_y|+\rangle = \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right);|+\rangle = \frac{i\hbar}{2}|-\rangle; \qquad S_y|-\rangle = \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right);|-\rangle = -\frac{i\hbar}{2}|+\rangle;$$

So the expectation values are

$$\begin{aligned}
\langle S_x\rangle &= \langle+|S_x|+\rangle = \langle+|\frac{\hbar}{2}|-\rangle = \frac{\hbar}{2}\langle+|-\rangle = 0\\
\langle S_y\rangle &= \langle+|S_y|+\rangle = \langle+|\frac{i\hbar}{2}|-\rangle = -i\frac{\hbar}{2}\langle+|-\rangle = 0\\
\left\langle S_x^2\right\rangle &= \langle+|S_x^2|+\rangle = \langle+|S_xS_x|+\rangle = \langle+|S_x\frac{\hbar}{2}|-\rangle = \frac{\hbar}{2}\langle+|\frac{\hbar}{2}|+\rangle = \frac{\hbar^2}{4}\langle+|-\rangle = \frac{\hbar^2}{4}\\
\left\langle S_y^2\right\rangle &= \langle+|S_y^2|+\rangle = \langle+|S_yS_y|+\rangle = \langle+|S_yi\frac{\hbar}{2}|-\rangle = \frac{i\hbar}{2}\langle+|\frac{-i\hbar}{2}|+\rangle = -i^2\frac{\hbar^2}{4}\langle+|-\rangle = \frac{\hbar^2}{4}
\end{aligned}$$

Since $[S_x, S_y] = i\hbar S_z$ and $|\langle[S_x,S_y]\rangle|^2 = \langle[S_x,S_y]\rangle\langle[S_x,S_y]\rangle^*$ we can write

$$\langle[S_x,S_y]\rangle = \langle i\hbar S_z\rangle = i\hbar\langle+|S_z|+\rangle = i\hbar\langle+|\frac{\hbar}{2}|+\rangle = i\frac{\hbar^2}{2}; \qquad \langle[S_x,S_y]\rangle^* = -i\frac{\hbar^2}{2};$$

The dispersion in S_x and S_y can be calculated as

$$\left\langle(\Delta S_x)^2\right\rangle \equiv \left\langle S_x^2\right\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}; \qquad \left\langle(\Delta S_x)^2\right\rangle \equiv \left\langle S_x^2\right\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4};$$

Thus finally

$$\begin{aligned}
\left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle &\geq \frac{1}{4}|\langle[S_x,S_y]\rangle|^2\\
\frac{\hbar^2}{4}\cdot\frac{\hbar^2}{4} &\geq \frac{1}{4}\left(i\frac{\hbar^2}{2}\right)\left(-i\frac{\hbar^2}{2}\right)\\
\frac{\hbar^4}{16} &\geq \frac{\hbar^4}{16}
\end{aligned}$$

Which is true as required. □

(b) Check the uncertainity relation with $A \to S_x, B \to S_y$ for the S_x+ State

3.2.3. Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the unertainity product $\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle$. Verify explicitly that the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

Solution:

Let the linear combination that maximizes the Uncertainity product be $p\,|+\rangle + q\,|-\rangle$. Since we know that the coefficients are complex in general and that the overall phase is immaterial, we can take p qnd q such that $p = r$ and $q = se^{i\delta}$ where r, s, δ are real numbers.

$$|\alpha\rangle = r\,|+\rangle + se^{i\delta}\,|-\rangle \qquad \leftarrow DC \rightarrow \qquad \langle\alpha| = \langle+|\,r + \langle-|\,se^{-i\delta}$$

Since Operator $S_x \equiv \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)$ and $S_y \equiv \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)$; we can find the expectation value

$$S_x\,|\alpha\rangle = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)\left(r\,|+\rangle + se^{i\delta}\,|-\rangle\right) = \frac{\hbar}{2}(se^{i\delta}\,|+\rangle + r\,|-\rangle)$$

$$\begin{aligned}
\langle S_x\rangle = \langle\alpha|S_x|\alpha\rangle &= \Big[\langle+|\,r + \langle-|\,se^{-i\delta}\Big]\frac{\hbar}{2}(se^{i\delta}\,|+\rangle + r\,|-\rangle) \\
&= \frac{\hbar}{2}\left\{rse^{i\delta} + rse^{-i\delta}\right\} \\
&= \frac{\hbar}{2}rs\left\{e^{i\delta} + e^{-i\delta}\right\} \\
&= \frac{\hbar}{2}rs2\cos(\delta) = \hbar rs\cos\delta
\end{aligned}$$

Also we can calculate the expectation value of S_x^2 which is

$$\begin{aligned}
\langle S_x^2\rangle = \langle\alpha|S_xS_x|\alpha\rangle = \langle\alpha|\,S_x\left(\frac{\hbar}{2}(se^{i\delta}\,|+\rangle + r\,|-\rangle)\right) \\
&= \Big[\langle+|\,r + \langle-|\,se^{-i\delta}\Big]\frac{\hbar^2}{4}(r\,|+\rangle + se^{i\delta}\,|-\rangle) \\
&= \frac{\hbar^2}{4}(r^2+s^2) = \frac{\hbar^2}{4} \qquad \text{(By normalization condition)}
\end{aligned}$$

Which can be use to calculate the dispersion of S_x as

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - \hbar^2r^2s^2\cos^2(\delta) = \frac{\hbar^2}{4}\left(1 - 4r^2s^2\cos^2(\delta)\right)$$

By similar procedure we can calculate $\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}(1 - 4r^2s^2\sin^2(\delta)$. So their product is

$$\begin{aligned}
\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle &= \frac{\hbar^2}{4}\left(1 - 4r^2s^2\cos^2(\delta)\right)\cdot\frac{\hbar^2}{4}\left(1 - 4r^2s^2\sin^2(\delta)\right) \\
&= \frac{\hbar^4}{16}(1 - 4r^2s^2\sin^2(\delta) - 4r^2s^2\cos^2(\delta) + 16r^4s^4\sin^2(\delta)\cos^2(\delta)) \\
&= \frac{\hbar^2}{16}(1 - 4r^2s^2 + 16r^4s^44\sin^2(\delta)\cos^2(\delta)) \\
&= \frac{\hbar^2}{16}(1 - 4r^2s^2 + 4r^4s^4\sin^2(2\delta))
\end{aligned}$$

Since r and s are constrained by normalization as $s = \sqrt{1-r^2}$. The two parameters for the variation of the product is δ and r (or s). The since $\sin^2(2\delta)$ can attain the maximum value of 1 whhich gives

$\sin^2(2\delta) = 1; \Rightarrow 2\delta = \frac{\pi}{2} \Rightarrow \delta = \frac{\pi}{4}$. So the uncertainity product reduces to

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle = \frac{\hbar^2}{16}(1 - 4r^2 s^2 + 4r^4 s^4)$$
$$= \frac{\hbar^2}{16}\left(1 - 2r^2 s^2\right)^2$$

The maximum value of this expression occurs when $2r^2s^2$ is the minimum, which by inspection is 0 at $r = 0$. Using this value $r = 0$ in normalization condition $r^2 + s^2 = 1$ gives $s = \pm 1$. So the linear combination we started reduces to

$$|\alpha\rangle = 0\,|+\rangle \pm e^{i\frac{\pi}{4}}\,|-\rangle = \left(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right)\left|-\right\rangle$$

□

3.2.4. Show that either $[A, B] = 0$ or $[B, C] = 0$ is sufficient for $\langle c'|a'\rangle$ to be
Solution:
Let the common eigenket of compatible operators A, B be $|a', b'\rangle$. Since they are observable the set of these eigenkets form a complete set let them be $|a', b'\rangle, |a'', b''\rangle \cdots |a^n, b^n\rangle$ for n state (dimensional) system. In the first way of invividually measuring the outcomes of B observables the total probability of observing $\left|c^1\right\rangle$ state is then

$$\left|\left\langle c^1\middle|a^1\right\rangle\right|^2 = \sum_i \left|\left\langle c^i\middle|a^i, b^i\right\rangle\right|^2 \left|\left\langle a^i, b^i\middle|s\right\rangle\right|^2$$

□

3.3 Homework Three

3.3.1. Using the rules of bra-ket algebra, prove or evaluate the following:

(a) $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, where X and Y are operators
Solution:
The definition of trace of an operator is $tr(A) = \sum_{a'} \langle a'|A|a'\rangle$. Using this definition for operator XY we get

$$\begin{aligned}
\mathrm{tr}(XY) &= \sum_{a'} \langle a'|XY|a'\rangle && (\text{ Definition}) \\
&= \sum_{a'}\sum_{a''} \langle a'|X|a''\rangle\,\langle a''|Y|a'\rangle && (\textstyle\sum_{a''} |a''\rangle\langle a''| = 1) \\
&= \sum_{a'}\sum_{a''} \langle a''|Y|a'\rangle\,\langle a'|X|a''\rangle && (\text{Complex number commute}) \\
&= \sum_{a''} \langle a''|YX|a''\rangle && (\textstyle\sum_{a'} |a'\rangle\langle a'| = 1) \\
&= \mathrm{tr}(YX) && (\text{By definition })
\end{aligned}$$

Thus $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ as required □

(b) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators.
Solution:

Let $|\alpha\rangle$ be any arbitrary ket.

$$\text{Let } Y|\alpha\rangle = |\gamma\rangle \qquad \leftarrow DC \rightarrow \qquad \langle\gamma| Y^\dagger = \langle\gamma|$$

Using this fact and operating the arbitrary $|\alpha\rangle$ by the operator XY we get,

$$\begin{aligned} XY|\alpha\rangle &= X|\gamma\rangle && (\because Y|\alpha\rangle = |\gamma\rangle \text{ by assumption}) \\ \langle\alpha|(XY)^\dagger &= \langle\gamma| X^\dagger && (\because \text{ Taking DC on both sides}) \\ \langle\alpha|(XY)^\dagger &= \langle\alpha| Y^\dagger X^\dagger && (\because \langle\gamma| = \langle\alpha| Y^\dagger \,) \end{aligned}$$

Which implies $(XY)^\dagger = X^\dagger Y^\dagger$ □

(c) $\exp(if(A)) = ?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known.
Solution:
Assuming the function can be written as $e^X = 1 + f(X) + \frac{f^2(X)}{2!} + \frac{f^3(X)}{3!} + \cdots$, where X is an operator in the ket space. We have

$$e^{if(A)} = \sum_{a'} e^{if(A)} |a'\rangle\langle a'| \qquad \left(\because \sum_{a'} |a'\rangle\langle a'|\right)$$

Here $|a'\rangle$ are the eigenkets of the operator A as it is given to be a Hermitian operator. Using the expansion for $e^{if(A)}$ we get,

$$\begin{aligned} e^{if(A)} &= \sum_{a'} \left(1 + f(X) + \frac{f^2(X)}{2!} + \frac{f^3(X)}{3!} + \cdots\right) |a'\rangle\langle a'| && \left(\because \sum_{a'} |a'\rangle\langle a'|\right) \\ &= \sum_{a'} \left(|a'\rangle + f(A)|a'\rangle + \frac{1}{2!} f^2(A)|a'\rangle + \cdots\right) \langle a'| && (\because X(|\alpha\rangle\langle\beta|) = (X|\alpha\rangle)\langle\beta|) \\ &= \sum_{a'} \left(|a'\rangle + f(a')|a'\rangle + \frac{1}{2!} f^2(a')|a'\rangle + \cdots\right) \langle a'| && (\because f(X)|a'\rangle = f(a')|a'\rangle \text{ for Hermitian } X) \\ &= \sum_{a'} \left(1 + f(a') + \frac{1}{2!} f^2(a') + \cdots\right) |a'\rangle\langle a'| && (\because (a|\alpha\rangle)\langle\beta| = a(|\alpha\rangle\langle\beta|)) \\ &= \sum_{a'} e^{f(a')} |a'\rangle\langle a'| \end{aligned}$$

Which is the required form for the operator $e^{f(A)}$. □

3.3.2. A spin 1/2 system is known tobe in an eigenstate of $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$ with eigenvalue $\hbar/2$, where $\hat{\boldsymbol{n}}$ is a unit vector lying in the xz-plane that makes and angle γ with the positive z-axis.

(a) Suppose S_x is measured. What is the probability of getting $\hbar/2$
Solution:
For a two state system the general stae of system can be represented as $|\hat{\boldsymbol{n}};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$, where α is the polar angle and β is the azimuthal angle. For this problem the polar angle is $\alpha = 0$ and azimuthal angle is $\beta = \gamma$. So the given system and $|S_x;+\rangle$ states are

$$|\hat{\boldsymbol{n}};+\rangle = \sin\frac{\gamma}{2}|+\rangle + \cos\frac{\gamma}{2}|-\rangle\,; \quad |S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

Since by definition the probablility of measuring any state that is known to be in $|beta\rangle$ in a state $|\alpha\rangle$ is given by $|\langle\alpha|\beta\rangle|^2$. So the probability of measuring $|S_x;+\rangle$ state when the system is known to be in $|\hat{\boldsymbol{n}};+\rangle$ stae is

$$\begin{aligned}
|\langle S_x;+|\hat{\boldsymbol{n}};+\rangle|^2 &= \left|\left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right)\left(\sin\frac{\gamma}{2}|+\rangle + \cos\frac{\gamma}{2}|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sin\frac{\gamma}{2} + \frac{1}{\sqrt{2}}\cos\frac{\gamma}{2}\right|^2 \\
&= \frac{1}{2}\sin^2\frac{\gamma}{2} + 2\frac{1}{\sqrt{2}}\sin\frac{\gamma}{2}\frac{1}{\sqrt{2}}\cos\frac{\gamma}{2} + \frac{1}{2}\cos^2\frac{\gamma}{2} \\
&= \frac{1}{2}(1+\sin\gamma)
\end{aligned}$$

So the probability of measuring the $|\hat{\boldsymbol{n}}\rangle$ state in $|S_x;+\rangle$ state is $(1+\sin\gamma)/2$. □

(b) Evaluate the dispersion in S_x –that is $\left\langle (S_x - \left\langle {S_x}^2 \right\rangle)\right\rangle$
Solution:
The S_x operator is $S_x = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)$. The result of S_x state operated on the system at $|\hat{\boldsymbol{n}}\rangle$ is

$$S_x|\hat{\boldsymbol{n}}\rangle = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)(\sin\frac{\gamma}{2}|+\rangle + \cos\frac{\gamma}{2}|-\rangle) = \frac{\hbar}{2}\cos\frac{\gamma}{2}|+\rangle + \frac{\hbar}{2}\sin\frac{\gamma}{2}|-\rangle$$

And the dual correspondence of the state $|\hat{\boldsymbol{n}}\rangle$ is $\langle\hat{\boldsymbol{n}}| = \sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|$. So the expectation value of S_x is

$$\langle S_x\rangle = \langle\hat{\boldsymbol{n}}|S_x|\hat{\boldsymbol{n}}\rangle = \left(\sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|\right)\left(\frac{\hbar}{2}\cos\frac{\gamma}{2}|+\rangle + \frac{\hbar}{2}\sin\frac{\gamma}{2}|-\rangle\right) = \frac{\hbar}{2}\left(2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\right) = \frac{\hbar}{2}\sin\gamma$$

Also the expectation value of operator S_x^2 is

$$\begin{aligned}
\left\langle S_x^2\right\rangle = \langle\hat{\boldsymbol{n}}|S_xS_x|\hat{\boldsymbol{n}}\rangle &= \left(\sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|\right)\left(\frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)\right)\left(\frac{\hbar}{2}\cos\frac{\gamma}{2}|+\rangle + \frac{\hbar}{2}\sin\frac{\gamma}{2}|-\rangle\right) \\
&= \left(\sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|\right)\left(\frac{\hbar^2}{4}\left(\sin\frac{\gamma}{2}|+\rangle + \cos\frac{\gamma}{2}|-\rangle\right)\right) \\
&= \frac{\hbar^2}{4}\left(\sin^2\frac{\gamma}{2} + \cos^2\frac{\gamma}{2}\right) \\
&= \frac{\hbar^2}{4}
\end{aligned}$$

Now the dispersion by definition is

$$\left\langle\Delta S_x^2\right\rangle \equiv \left\langle S_x^2\right\rangle - (\langle S_x\rangle)^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\sin\gamma\right)^2 = \frac{\hbar^2}{4}\left(1-\sin^2\gamma\right) = \frac{\hbar^2}{4}\cos^2\gamma$$

Which gives the dispersion in measurement of S_x of the system in $|\hat{\boldsymbol{n}}\rangle$. □

3.3.3. Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent withthe general relation $U = \sum_r \left|b^{(r)}\right\rangle\!\left\langle a^{(r)}\right|$

Solution:
The states $|S_x;\pm\rangle$ in the $|S_z;\pm\rangle \equiv |\pm\rangle$ state is given by $|S_x;\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$. Since we know the transformation matrix form is

$$\begin{bmatrix}\langle S_x;+|+\rangle & \langle S_x;+|-\rangle \\ \langle S_x;-|+\rangle & \langle S_x;-|-\rangle\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}(\langle +| + \langle -|)|+\rangle & (\langle +| + \langle -|)|-\rangle \\ (\langle +| - \langle -|)|+\rangle & (\langle +| - \langle -|)|-\rangle\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}$$

Let $|p\rangle = a\,|+\rangle + a\,|-\rangle$ in the old S_z basis. such that $a = \langle +|p\rangle$ and $b = \langle -|p\rangle$. This ket is transformed into

$$Mp = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} \equiv \frac{1}{\sqrt{2}}(a+b)\,|+\rangle + \frac{1}{\sqrt{2}}(a-b)\,|-\rangle \tag{3.2}$$

$$= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)a + \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)b \tag{3.3}$$

$$= \frac{1}{\sqrt{2}}((|+\rangle + |-\rangle)\,\langle +|p\rangle + \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\,\langle -|p\rangle) \tag{3.4}$$

$$= (|S_x;+\rangle\,\langle +| + |S_x;-\rangle\,\langle -|)\,|p\rangle \tag{3.5}$$

Which is in the form of $\sum |b^r\rangle\langle a^r|$. □

3.3.4. Prove that $\langle \boldsymbol{x}\rangle \to \langle \boldsymbol{x}\rangle + d\boldsymbol{x}'$, $\langle \boldsymbol{p}\rangle \to \langle \boldsymbol{p}\rangle$ under infinitesimal translation.
Solution:
Since given

$$[\boldsymbol{x}, \mathcal{T}(d\boldsymbol{x})] = d\boldsymbol{x}; \Rightarrow \boldsymbol{x}\mathcal{T}(d\boldsymbol{x}) - \mathcal{T}(d\boldsymbol{x})\boldsymbol{x} = d\boldsymbol{x}; \qquad \boldsymbol{x}\mathcal{T}(d\boldsymbol{x}) = d\boldsymbol{x} + \mathcal{T}(d\boldsymbol{x})\boldsymbol{x}$$

Let the state of system under translation be $|\beta\rangle = \mathcal{T}(d\boldsymbol{x})\,|\alpha\rangle$, thus $\langle\beta| = \langle\alpha|\,\mathcal{T}^\dagger(d\boldsymbol{x})$. Now the expectation value of system before translation is $\langle \boldsymbol{x}\rangle = \langle\alpha|\boldsymbol{x}|\alpha\rangle$. The expectation value after translation is

$$\begin{aligned}
\langle \boldsymbol{x}\rangle &= \langle\beta|\boldsymbol{x}|\beta\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x})\boldsymbol{x}\mathcal{T}(d\boldsymbol{x})|\alpha\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x})(d\boldsymbol{x} + \mathcal{T}(d\boldsymbol{x})\boldsymbol{x})|\alpha\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x}) + \mathcal{T}^\dagger(d\boldsymbol{x})\mathcal{T}(d\boldsymbol{x})\boldsymbol{x}|\alpha\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x}) + \boldsymbol{x}|\alpha\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x})|\alpha\rangle + \langle\alpha|\boldsymbol{x}|\alpha\rangle\\
&= d\boldsymbol{x} + \langle \boldsymbol{x}\rangle
\end{aligned}$$

So the expectation value of position after translation is $\langle \boldsymbol{x}\rangle + d\boldsymbol{x}$.

Similarly for momentum

$|\beta\rangle = \mathcal{T}(d\boldsymbol{x})\,|\alpha\rangle$, thus $\langle\beta| = \langle\alpha|\,\mathcal{T}^\dagger(d\boldsymbol{x})$. Now the expectation value of momentum before translation is $\langle \boldsymbol{p}\rangle = \langle\alpha|\boldsymbol{p}|\alpha\rangle$. The expectation value after translation is

$$\begin{aligned}
\langle \boldsymbol{p}\rangle &= \langle\beta|\boldsymbol{p}|\beta\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x})\boldsymbol{p}\mathcal{T}(d\boldsymbol{x})|\alpha\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x})(0 + \mathcal{T}(d\boldsymbol{x})\boldsymbol{p})|\alpha\rangle\\
&= \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x})\mathcal{T}(d\boldsymbol{x})\boldsymbol{p}|\alpha\rangle\\
&= \langle\alpha|\boldsymbol{p}|\alpha\rangle
\end{aligned}$$

So the expectation value of system after translation is still $\langle \boldsymbol{p}\rangle$. □

3.4 Homework Four

3.4.1. Some authors define an *operator* to be real when every member of its matrix elements $\langle b'|A|b''\rangle$ is real in some representation. Is this concept representation independent? That is, do the matrix elements

remain real even if some basis other than $\{|b'\rangle\}$ is used? Check your assertion using x and p_x.
Solution:
Let some other basis $|a'\rangle$ be used to represent the matrix then the new basis is related to the old basis by the transformation $|a'\rangle = U|b'\rangle$ where U is some unitary operator.

$$|a'\rangle = U|b'\rangle; \quad \Rightarrow \langle a'| = \langle b'| U^\dagger = \langle b'| U^{-1}$$

The matrix elements in this new basis then become

$$\langle a'|A|a''\rangle = \langle b'|U^{-1}AU|b'\rangle$$

If this has to remain real in the old $|b'\rangle$ basis then it must equal to the old matrix element

$$\langle b'|U^{-1}AU|b'\rangle = \langle b'|A|b''\rangle; \qquad \Rightarrow U^{-1}AU = A; \qquad \Rightarrow AU = UA; \qquad \Rightarrow [U, A] = 0$$

But it is not necessary that the operators U and A commute i.e.,$[U, A] = 0$. Thus the matrix element of an operator may not remain real in a different basis if it is real in one basis.

Checking this assertion with x and p_x. We know that operator x is hermitian in x basis so that the eigenvalues of x in position $|x'\rangle$ basis are real. Which means the the matrix elements $\langle x'|x|x''\rangle = x''\langle x'|x''\rangle = x''\delta(x'-x'')$ are all real because x'' is real eigenvalue of hermitian operator of x.

Now the matrix elements of x operator in p basis are

$$\begin{aligned}\langle p'|x|p''\rangle &= \int \langle p'|x|x'\rangle\langle x'|p''\rangle\, dx' = \int x'\langle p'|x'\rangle\langle x'|p''\rangle \\ &= \frac{1}{2\pi\hbar}\int x' \exp\left(-\frac{ip'x}{\hbar}\right)\exp\left(\frac{ip''x}{\hbar}\right)dx = \frac{1}{2\pi\hbar}\int x' \exp\left(i\frac{(p''-p')x'}{\hbar}\right)dx'\end{aligned}$$

making substitution $t = p'' - p'$ and $y = x'/\hbar$

$$= \frac{1}{2\pi\hbar}\int \hbar y e^{ity}\hbar dy = \frac{\hbar}{2\pi}\int y e^{ity} dy$$

and using differential under integral sign $\frac{d}{dt}\int e^{ity}dy = \int iye^{ity}dy \Rightarrow \int ye^{ity}dy = \frac{1}{i}\frac{d}{dt}\int e^{ity}dy$ we can write the above expression as

$$\langle p'|x|p''\rangle = \frac{\hbar}{2\pi}\frac{1}{i}\frac{d}{dt}\int e^{ity}dy = \frac{\hbar}{2\pi i}\frac{d}{dt}\int e^{i(p''-p')y}dy = \frac{\hbar}{2\pi i}\frac{d}{dt}2\pi\delta(p''-p') = \frac{\hbar}{i}\frac{d}{dt}\delta(p''-p')$$

This value is clearly imaginary as delta function is purely real. This shows that although the matrix elements of operator x in position basis are real the elements are no longer real in momentum basis. □

3.4.2. (a) Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A|a'\rangle = a'|a'\rangle$. Evaluate $\langle b''|f(A)|b'\rangle$ when the transformation matrix from the a' basis to the b' basis is known.
Solution:
The matrix element for the transformation matrix are $\left\langle b^{(i)}\middle|a^{(j)}\right\rangle$ for $i, j \in \{1, 2\cdots N\}$ where N is the no of independent state of system. The given expression can be written as

$$\begin{aligned}\langle b''|f(A)|b'\rangle &= \sum_i \left\langle b''\middle|f(A)\middle|a^i\right\rangle\left\langle a^i\middle|b'\right\rangle && (\because \text{ Inserting } \sum_i \left|a^i\middle\rangle\middle\langle a^i\right| = 1) \\ &= \sum_i \left\langle b''\middle|f(a^i)\middle|a^i\right\rangle\left\langle a^i\middle|b'\right\rangle && (\because f(A)|a'\rangle = f(a')|a'\rangle) \\ &= \sum_i f(a^i)\left\langle b''\middle|a^i\right\rangle\left\langle a^i\middle|b'\right\rangle && (\because \langle\alpha|c|\beta\rangle = c\langle\alpha|\beta\rangle)\end{aligned}$$

Since all the matrix elements $\left\langle b''\middle|a^i\right\rangle$ and $\left\langle a^i\middle|b'\right\rangle = \left\langle b'\middle|a^i\right\rangle^*$ are known the expression is completely known. □

(b) Using the continuum analogue of the result obtained in (3.4.2a), evaluate $\langle \boldsymbol{p''}|F(r)|\boldsymbol{p'}\rangle$. Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, and z are *operators*.
Solution:
Since the position operators x, y and z are compatible operators (commutative i.e., $[x, y] = 0, [y, z] = 0$ and $[z, x] = 0$) we can represent the position eigenket as $|x', y', z'\rangle \equiv |\boldsymbol{r'}\rangle$. By problem (3.4.2a) above we get

$$\langle \boldsymbol{p''}|F(\boldsymbol{r})|\boldsymbol{p'}\rangle = \int_{-\infty}^{\infty} F(r') \langle \boldsymbol{p''}|\boldsymbol{r'}\rangle \langle \boldsymbol{r'}|\boldsymbol{p'}\rangle \, d^3\boldsymbol{r'}$$

But we know the wavefunction of momentum in position basis as

$$\langle \boldsymbol{p}|\boldsymbol{r}\rangle = e^{-\dfrac{i\boldsymbol{p}\cdot\boldsymbol{r}}{\hbar}} \qquad \Rightarrow \langle \boldsymbol{p''}|\boldsymbol{r'}\rangle = e^{-\dfrac{i\boldsymbol{p''}\cdot\boldsymbol{r'}}{\hbar}} \quad \text{and} \qquad \langle \boldsymbol{r'}|\boldsymbol{p'}\rangle = e^{-\dfrac{-i\boldsymbol{p'}\cdot\boldsymbol{r'}}{\hbar}}$$

Thus the expression becomes

$$\langle \boldsymbol{p''}|F(\boldsymbol{r})|\boldsymbol{p'}\rangle = \int_{-\infty}^{\infty} F(r') e^{-\dfrac{i(\boldsymbol{p'}-\boldsymbol{p''})\cdot\boldsymbol{r'}}{\hbar}} \, d^3\boldsymbol{r'}$$

This integral gives the matrix element of the position operator $F(\boldsymbol{r})$ in the momentum $\boldsymbol{p'}$ basis. □

3.4.3. The translation operator for a finite (spatial) displacement is given by

$$\mathfrak{T}(\boldsymbol{l}) = \exp\left(\frac{-i\boldsymbol{p}\cdot\boldsymbol{l}}{\hbar}\right),$$

where $\boldsymbol{p}$ is the momentum *operator*.

(a) Evaluate $[x_i, \mathfrak{T}(\boldsymbol{l})]$
Solution:
We can write the dot product of vectors $\boldsymbol{p}$ and displacement $\boldsymbol{l}$ as $\boldsymbol{p}\cdot\boldsymbol{l} = \sum_i p_i l_i$

$$[x_i, \mathfrak{T}(\boldsymbol{l})] = \left[x_i, \exp\left(\frac{-i\boldsymbol{p}\cdot\boldsymbol{l}}{\hbar}\right)\right] = i\hbar\frac{\partial}{\partial p_i}\exp\left(\frac{-i\sum_i p_i l_i}{\hbar}\right) = i\hbar l_i\left(\frac{-i}{\hbar}\right)\exp\left(-\frac{i\boldsymbol{p}\cdot\boldsymbol{l}}{\hbar}\right) = l_i\mathfrak{T}(\boldsymbol{l})$$

This gives the expression for $[x_i, \mathfrak{T}(\boldsymbol{l})]$. □

(b) Using (3.4.3a) (or otherwise), demonstrate how expectation value of $\langle \boldsymbol{x}\rangle$ changes under translation
Solution:
Let $|\alpha\rangle$ be any arbitrary position ket. Then the expectation value of for one of the component of position of the system (particle) is given by $\langle x_i\rangle = \langle\alpha|x_i|\alpha\rangle$. Let the position ket under translation be $|\beta\rangle \equiv \mathfrak{T}(\boldsymbol{l})\,|\alpha\rangle$. The dual correspondence of this ket is $\langle\beta| = \langle\alpha|\,\mathfrak{T}(\boldsymbol{l})^\dagger$. Now the expectation value under translation is

$$\langle\beta|x_i|\beta\rangle = \langle\alpha|\mathfrak{T}(\boldsymbol{l})^\dagger x_i \mathfrak{T}(\boldsymbol{l})|\alpha\rangle \tag{3.6}$$

But by the commutator relation (3.4.3a) we have

$$[x_i, \mathfrak{T}(\boldsymbol{l})] = l_i\mathfrak{T}(\boldsymbol{l}); \qquad \Rightarrow x_i\mathfrak{T}(\boldsymbol{l}) - \mathfrak{T}(\boldsymbol{l})x_i = l_i\mathfrak{T}(\boldsymbol{l})$$

Since we know that the translation operator is Unitary, $\mathfrak{T}(\boldsymbol{l})^\dagger\mathfrak{T}(\boldsymbol{l}) = 1$. Operating on both sides of this expression by $\mathfrak{T}(\boldsymbol{l})^\dagger$ we get

$$\begin{aligned}
\mathfrak{T}(\boldsymbol{l})^\dagger\{x_i\mathfrak{T}(\boldsymbol{l}) - \mathfrak{T}(\boldsymbol{l})x_i\} &= \mathfrak{T}(\boldsymbol{l})^\dagger l_i\mathfrak{T}(\boldsymbol{l}) \\
\Rightarrow \qquad \mathfrak{T}(\boldsymbol{l})^\dagger x_i\mathfrak{T}(\boldsymbol{l}) - \mathfrak{T}(\boldsymbol{l})^\dagger\mathfrak{T}(\boldsymbol{l})x_i &= l_i\mathfrak{T}(\boldsymbol{l})^\dagger\mathfrak{T}(\boldsymbol{l}) \\
\Rightarrow \qquad \mathfrak{T}(\boldsymbol{l})^\dagger x_i\mathfrak{T}(\boldsymbol{l}) &= x_i + l_i
\end{aligned}$$

Using this in (3.6) we get

$$\langle\beta|x_i|\beta\rangle = \langle\alpha|x_i + l_i|\alpha\rangle = \langle\alpha|x_i|\alpha\rangle + \langle\alpha|l_i|\alpha\rangle = \langle\alpha|x_i|\alpha\rangle + l_i$$

Now that we have found the expectation value of every component of $\boldsymbol{x}$ operator. The expression for this operator becomes

$$\langle\beta|x_i|\beta\rangle = \langle\alpha|x_i|\alpha\rangle + l_i; \qquad \Rightarrow \langle\boldsymbol{x}\rangle \xrightarrow{\mathfrak{T}(\boldsymbol{l})} \langle\boldsymbol{x}\rangle_{\text{old}} + \boldsymbol{l}$$

This gives the expectation value of position operator under translation. □

3.4.4. For a Gaussian wave packet, whose wave function is position space is given by

$$\langle x'|\alpha\rangle = \left[\frac{1}{\sqrt{d\sqrt{\pi}}}\right] \exp\left[ikx' - \frac{x'^2}{2d^2}\right]$$

(a) Verify $\langle p\rangle = \hbar k$ and $\langle p^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$

Solution:

The expectation value of momentum p in the state $|\alpha\rangle$ is given by $\langle p\rangle = \langle\alpha|p|\alpha\rangle$. But by completeness of the position basis kets we can write the state $|\alpha\rangle$ as

$$\langle p\rangle = \langle\alpha|p|\alpha\rangle = \int dx' \, \langle\alpha|x'\rangle \, \langle x'|p|\alpha\rangle$$

But the operator identity

$$\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

Enables us to write

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty} dx' \, \langle\alpha|x'\rangle \left(-i\hbar\frac{\partial}{\partial x'}\right) \langle x'|\alpha\rangle \\
&= \int_{-\infty}^{\infty} dx' \left[\frac{1}{\sqrt{d\sqrt{\pi}}}\right] \exp\left[-ikx' - \frac{x'^2}{2d^2}\right] \left(-i\hbar\frac{\partial}{\partial x'}\right) \left[\frac{1}{\sqrt{d\sqrt{\pi}}}\right] \exp\left[ikx' - \frac{x'^2}{2d^2}\right] \\
&= \frac{1}{d\sqrt{\pi}} \int_{-\infty}^{\infty} dx' \left(-i\hbar\left(ik - \frac{x'}{d^2}\right)\right) \exp\left(-\frac{x'^2}{d^2}\right) \\
&= \frac{1}{d\sqrt{\pi}} \left[\hbar k \int_{-\infty}^{\infty} dx' \exp\left(-\frac{x'^2}{d^2}\right) + \frac{i\hbar}{d^2} \int_{-\infty}^{\infty} x' \exp\left(-\frac{x'^2}{d^2}\right)\right] \\
&= \frac{1}{d\sqrt{\pi}} \left[\hbar k\sqrt{\pi} d + \frac{i\hbar}{d^2} 0\right] \\
&= \hbar k
\end{aligned}$$

Smilarly the expectation value of operator p^2 can be written as

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} dx' \langle \alpha | x' \rangle \left(-i\hbar \frac{\partial}{\partial x'} \right)^2 \langle x' | \alpha \rangle \\
&= \int_{-\infty}^{\infty} dx' \left[\frac{1}{\sqrt{d\sqrt{\pi}}} \right] \exp\left[-ikx' - \frac{x'^2}{2d^2} \right] \left(-\hbar^2 \frac{\partial^2}{\partial x'^2} \right) \left[\frac{1}{\sqrt{d\sqrt{\pi}}} \right] \exp\left[ikx' - \frac{x'^2}{2d^2} \right] \\
&= \frac{1}{d\sqrt{\pi}} \int_{-\infty}^{\infty} dx' \left(\frac{\hbar^2}{d^2} - \hbar^2 \left(ik - \frac{x'}{d^2} \right)^2 \right) \exp\left(-\frac{x'^2}{d^2} \right) \\
&= \frac{1}{d\sqrt{\pi}} \int_{-\infty}^{\infty} dx' \left(\frac{\hbar^2}{d^2} + \hbar^2 k^2 + \frac{2ik\hbar^2}{d^2} - \frac{\hbar^2 x'^2}{d^4} \right) \exp\left(-\frac{x'^2}{d^2} \right) \\
&= \frac{1}{d\sqrt{\pi}} \left[\left(\frac{\hbar^2}{d^2} + \hbar^2 k^2 \right) \int_{-\infty}^{\infty} \exp\left(-\frac{x'^2}{d^2} \right) dx' + \frac{2ik\hbar^2}{d^2} \int_{-\infty}^{\infty} x' \exp\left(-\frac{x'^2}{d^2} \right) dx' - \frac{\hbar^2}{d^4} \int_{-\infty}^{\infty} x'^2 \exp\left(-\frac{x'^2}{d^2} \right) dx' \right] \\
&= \frac{1}{d\sqrt{\pi}} \left[\left(\frac{\hbar^2}{d^2} + \hbar^2 k^2 \right) \sqrt{\pi} d + \frac{2ik\hbar^2}{d^2} 0 - \frac{\hbar^2}{d^4} \left(\frac{\sqrt{\pi} d^3}{2} \right) \right] \\
&= \frac{\hbar^2}{d^2} + \hbar^2 k^2 - \frac{\hbar^2}{2d^2} \\
&= \frac{\hbar^2}{2d^2} + \hbar^2 k^2
\end{aligned}$$

Thus the expectation values of the wavefunction is found as required. □

(b) Evaluate the expectation value of p and p^2 usig the momentum-space wave functions as well.
Solution:
For the momentum space wave functions we can write

$$\begin{aligned}
\langle p \rangle &= \int \langle \alpha | p | p' \rangle \langle p' | \alpha \rangle \, dp' = \int p' |\langle p' | \alpha \rangle|^2 dp' \\
&= \frac{d}{\hbar\sqrt{\pi}} \int p' \exp\left[-\frac{(p' - \hbar k)^2 d^2}{\hbar^2} \right] dp' \\
&= \frac{d}{\hbar\sqrt{\pi}} \left[\int p' \exp\left(-\frac{d^2}{\hbar^2} \right) dp' + \int p' \exp\left(\frac{(p - \hbar k)^2}{\hbar^2} \right) dp' \right] \\
&= \frac{d}{\hbar\sqrt{\pi}} \left[\frac{\hbar^2 k \sqrt{\pi}}{d} \right] = \hbar k
\end{aligned}$$

Now for the expectation value of the square of momentum operator.

$$\begin{aligned}
\langle p^2 \rangle &= \int \langle \alpha | p | p' \rangle \langle p' | \alpha \rangle \, dp' = \int p'^2 |\langle p' | \alpha \rangle|^2 dp' \\
&= \frac{d}{\hbar\sqrt{\pi}} \int p'^2 \exp\left[-\frac{(p' - \hbar k)^2 d^2}{\hbar^2} \right] dp' \\
&= \frac{d}{\hbar\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} \frac{\hbar^3}{d^3} + \frac{\hbar^3 k^2 \sqrt{\pi}}{d} \right) \\
&= \frac{\hbar^2}{2d^2} + \hbar^2 k^2
\end{aligned}$$

So the expectation value of the operators are the smae in the momentum state wave functions too.
□

3.5 Homework Five

3.5.1. (a) Prove the following

i. $\langle p'|x|\alpha\rangle = i\hbar\dfrac{\partial}{\partial p'}\langle p'|\alpha\rangle$

ii. $\langle \beta|x|\alpha\rangle = \displaystyle\int dp' \phi_\beta^*(p') i\hbar \frac{\partial}{\partial p'}\phi_\alpha(p')$,

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.

Solution:

We know

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right); \text{ and } \int_{-\infty}^{\infty} e^{i(t-t')x}dx = 2\pi\delta(t-t')$$

With the help of these two relations we can simplify the quantity we want as

$$\begin{aligned}
\langle p'|x|\alpha\rangle &= \int dx' \langle p'|x|x'\rangle \langle x'|\alpha\rangle && (\because \int dx' |x'\rangle\langle x'| = 1)\\
&= \int x' \langle p'|x'\rangle \langle x'|\alpha\rangle \, dx' && (\because \langle p'|x|x'\rangle = x\langle p'|x'\rangle)\\
&= \int dp'' \int x' \langle p'|x'\rangle \langle x'|p''\rangle \langle p''|\alpha\rangle \, dx' && (\because \int dp'' |p''\rangle\langle p''| = 1)\\
&= \int dp'' \int x' \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)\cdot\frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{ip''x'}{\hbar}\right)\langle p''|\alpha\rangle \, dx'\\
&= \frac{1}{2\pi\hbar}\int dp'' \int x' \exp\left(\frac{i(p'-p'')x'}{\hbar}\right)\langle p''|\alpha\rangle \, dx'
\end{aligned}$$

We can use integral under differential sign to evaluate the dx' integral as

$$\frac{d}{dp'}\int \exp(i(p'-p'')x')dx' = \int x' \exp(i(p'-p'')x')dx'$$

Using ths in the dx' integral above we get

$$\begin{aligned}
&= \frac{1}{2\pi\hbar}\int dp'' \frac{\hbar^2}{-i}\frac{\partial}{\partial p'}\int \exp\left(\frac{i(p'-p'')x'}{\hbar}\right)\langle p''|\alpha\rangle \, dx'\\
&= \frac{1}{2\pi\hbar}\int dp'' \frac{\hbar^2}{-i}\frac{\partial}{\partial p'} 2\pi\delta(p'-p'')\langle p''|\alpha\rangle\\
&= i\hbar\langle p'|\alpha\rangle && (\because \int f(x)\delta(x-x')dx = f(x')
\end{aligned}$$

This gives us the requied result.

$$\langle \beta|x|\alpha\rangle = \int dp' \langle \beta|p'\rangle \langle p'|x|\alpha\rangle \tag{3.7}$$

The result above is $\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$ Substuting this in (3.7) we get

$$\langle \beta|x|\alpha\rangle = \int dp' \langle \beta|p'\rangle i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

Writing $\langle \beta|p'\rangle = \phi_\beta^*(p')$ and $\langle p'|\alpha\rangle = \phi_\alpha(p')$ we get

$$\langle \beta|x|\alpha\rangle = \int dp' \phi_\beta^*(p') i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$$

This is the requied expression. □

(b) What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right)$$

where x is the position operator and Ξ is some number with the unit of momentum? Justify your answer.
Solution:
In the position eigenbasis the position translation operator is $\sqcup(l) = \exp\left(\frac{ipl}{\hbar}\right)$ where l is a constant of unit of lenght and p is the momentum operator.
We have here the roles of operator x and p changed and l and Ξ changed. Which suggests that this operator function can works as a momentum translation operator in momentum eigenbasis. □

3.5.2. If the Hamiltonian H is given as

$$H = H_{11}\,|1\rangle\langle 1| + H_{22}\,|2\rangle\langle 2| + H_{12}\,|1\rangle\langle 2|$$

What principle is violated? Illustrate your point by explicitly attempting to solve the most general time-dependent problem using an illegal Hamiltonian of this kind. (Assume $H_{11} = H_{22} = 0$ for simplicity.)
Solution:
For a operator to be a valid Hamiltonian it has to be a Hermitian operator. We can check if this is a Hermitian operator.

$$H^\dagger = H_{11}^*\,|1\rangle\,\langle 1| + H_{22}^*\,|2\rangle\,\langle 2| + H_{12}^*\,|1\rangle\,\langle 2| = H_{11}\,|1\rangle\,\langle 1| + H_{22}\,|2\rangle\,\langle 2| + H_{12}\,|1\rangle\,\langle 2|$$

Since $H^\dagger \neq H$ the given hamiltonian is clearly not Hermitian. So this operator the energy eigenkets won't be real. Also, the time translation operator $\mathcal{U}(t) = \exp\left(-\frac{iHt}{\hbar}\right)$ will not be unitary which would make the time evolved states not conserve the inner product so, it violates the principle of probability violation.

Setting $H_{11} = H_{22} = 0$ the Hamiltonian becomes $H = H_{12}\,|1\rangle\langle 2|$. Lets check the unitary property of the unitary operator

$$\mathcal{U}^\dagger(t)\mathcal{U}(t) = \exp\left(\frac{iH^\dagger t}{\hbar}\right)\cdot\exp\left(-\frac{iHt}{\hbar}\right) = \exp\left(\frac{i(H^\dagger - H)t}{\hbar}\right)$$

For the operator to remain unitary, the exponential should be zero but since $H^\dagger \neq H$ the exponent will be nonzero and it violates the principle that the time evolution operator si unitary. □

3.5.3. Let $|a'\rangle$ and $|a''\rangle$ be eigenstates of a Hermitian operator A with eigenvalues a' and a'', respectively ($a' \neq a''$). The Hamiltonian operator is given by

$$H = |a'\rangle\,\delta\,\langle a''| + |a''\rangle\,\delta\,\langle a'|$$

where δ is just a real number.

(a) Clearly, $|a'\rangle$ and $|a''\rangle$ are not eigenstates of teh Hamiltonian. Write down the eigenstates of the Hamiltonian. WHat are their energy Eigenvalues?
Solution:
Let the energy eigenket of this hamiltonian operator be $|\alpha\rangle = p\,|a'\rangle + q\,|a''\rangle$. And E be the energy eigen values. So operating by H on this state leads to

$$\begin{aligned} H\,|\alpha\rangle &= (|a'\rangle\,\delta\,\langle a''| + |a''\rangle\,\delta\,\langle a'|)(p\,|a'\rangle + q\,|a''\rangle) \\ &= \delta q\,|a'\rangle + \delta p\,|a''\rangle \end{aligned}$$

If this is to be the energy eigenstate then it should equal $E|\alpha\rangle = Ep|a'\rangle + Eq|a''\rangle$. Since $|a'\rangle$ and $|a''\rangle$ are orthogonal states, the coefficient comparision leads to

$$Ep = \delta q; \qquad \Rightarrow p = \frac{\delta q}{E}$$
$$Eq = \delta p; \qquad \Rightarrow Eq = \delta\frac{\delta q}{E}; \qquad \Rightarrow E = \pm\delta$$

So the energy eigenvalues are $E = \pm\delta$. Also since we require the eigenstate be normalized we require $p^2 + q^2 = 1$. This results in

$$\frac{\delta^2 q^2}{E^2} + q^2 = 1; \qquad \Rightarrow p = \frac{1}{\sqrt{2}}, \quad q = \pm\frac{1}{\sqrt{2}}$$

So the requried energy eigenkets are

$$|\alpha_+\rangle = \frac{1}{\sqrt{2}}(|a'\rangle + |a''\rangle); \qquad |\alpha_-\rangle = \frac{1}{\sqrt{2}}(|a'\rangle - |a''\rangle) \tag{3.8}$$

Where $|\alpha_+\rangle$ is the eigenket corresponding to eigenvalue $+\delta$ and $|\alpha_-\rangle$ is the eigenket corresponding to eigenvalue $-\delta$ □

(b) Suppose the system is known to be in the state $|a'\rangle$ at $t = 0$. Write down the state vector of Schrodinger picture for $t > 0$.
Solution:
The time evolution operator is $\mathcal{U}(t) = \exp\left(-\frac{iHt}{\hbar}\right)$. Since $|a'\rangle$ are not the eiergy eigenkets, we can write them in terms of the eigenkets of Hamiltonian operator. From (3.8) we can add and subtract the two energy eigenkets to find

$$|a'\rangle = \frac{1}{\sqrt{2}}(|\alpha_+\rangle + |\alpha_-\rangle) \qquad\qquad |a''\rangle = \frac{1}{\sqrt{2}}(|\alpha_+\rangle - |\alpha_-\rangle)$$

Application of time evolution operator to $|a'\rangle$ leads to

$$\mathcal{U}(t)|a'\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|a'\rangle = \exp\left(-\frac{iHt}{\hbar}\right)\frac{1}{\sqrt{2}}(|\alpha_+\rangle + |\alpha_-\rangle = \frac{1}{\sqrt{2}}e^{-i\delta\frac{t}{\hbar}}|\alpha_+\rangle + \frac{1}{\sqrt{2}}e^{i\delta\frac{t}{\hbar}}|\alpha_-\rangle$$

Again the application of (3.8) we can convert back to the basis states given

$$\mathcal{U}(t)|a'\rangle = \frac{1}{2}e^{-i\delta\frac{t}{\hbar}}(|a'\rangle + |a''\rangle) + \frac{1}{2}e^{i\delta\frac{t}{\hbar}}(|a'\rangle - |a''\rangle) = \frac{1}{2}(\underbrace{e^{-i\frac{\delta t}{\hbar}} + e^{i\frac{\delta t}{\hbar}}}_{2\cos\left(\frac{\delta t}{\hbar}\right)})|a'\rangle + \frac{1}{2}(\underbrace{e^{-i\frac{\delta t}{\hbar}} - e^{i\frac{\delta t}{\hbar}}}_{2i\sin\left(\frac{\delta t}{\hbar}\right)})|a''\rangle$$

Euler identity can be used to convert the complex exponentials to sines and cosies, which give

$$\mathcal{U}(t)|a'\rangle = \cos\left(\frac{\delta t}{\hbar}\right)|a'\rangle + i\sin\left(\frac{\delta t}{\hbar}\right)|a''\rangle \tag{3.9}$$

This gives the time evolution of state $|a'\rangle$ under this hamiltonian. □

(c) What is the probability for finding the system in $|a''\rangle$ for $t > 0$ if the system is known to be in the state $|a'\rangle$ at $t = 0$?
Solution:
The probability of finting the system knon to be in $|a'\rangle$ at a later time $t > 0$ is given by $|\langle a''|\mathcal{U}(t)|a'\rangle|^2$ which can be evauated using (3.9)

$$P = |\langle a''|\mathcal{U}(t)|a'\rangle|^2 = \left|\langle a''|\left[\cos\left(\frac{\delta t}{\hbar}\right)|a'\rangle + i\sin\left(\frac{\delta t}{\hbar}\right)|a''\rangle\right]\right|^2 = \left|i\sin\left(\frac{\delta t}{\hbar}\right)\right|^2 = \sin^2\left(\frac{\delta t}{\hbar}\right)$$

So the probability of finding the $|a'\rangle$ to be at $|a''\rangle$ at a later time is the oscillating function. The physical situation corresponding to this problem is a Neutrino oscillation. □

3.5.4. Show

$$\langle p'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\int_{-\infty}^{\infty} dx' \exp\left(\frac{-ip'x'}{\hbar} + ikx' - \frac{x'^2}{2d^2}\right) = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right].$$

Solution:
Considering the factor inside the exponential

$$\frac{-ip'x'}{\hbar} + ikx' - \frac{x'^2}{2d^2} = -\frac{1}{2d^2}\left(x'^2 - 2d^2\left(ik - \frac{ip'x}{\hbar}\right)x'\right)$$

If we let the constant terms $t = d^2\left(ik - \frac{ip'x}{\hbar}\right)$ then in the exponential we get

$$\frac{-1}{2d^2}\left(x'^2 - 2tx'\right) \xrightarrow{\text{Completion of square}} \frac{-1}{2d^2}\left((x'-t)^2 - t^2\right)$$

With this the integral becomes

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x'^2}{2d^2}\right)\cdot\exp\left(\frac{t^2}{2d^2}\right)dx' = \exp\left(-\frac{t^2}{2d^2}\right)\int_{-\infty}^{\infty} \exp\left\{-\left(\frac{x'}{\sqrt{2}d}\right)^2\right\}dx'$$

This integral is a standard gamma function whose value is

$$\int_{-\infty}^{\infty} \exp\left\{-\left(\frac{x'}{\sqrt{2}d}\right)^2\right\}dx' = 2\int_{0}^{\infty} \exp\left\{-\left(\frac{x'}{\sqrt{2}d}\right)^2\right\}dx' = \frac{\sqrt{\pi}}{2}\cdot 2\sqrt{2}d$$

Using this in our original equation we get

$$\langle p'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\exp\left(-\frac{t^2}{2d^2}\right)\sqrt{2\pi}d$$

We can substitute back the variable t back to get

$$\langle p'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\exp\left(-\frac{t^2}{2d^2}\right)\sqrt{2\pi}d = \frac{1}{\sqrt{\hbar}}\left(\frac{\sqrt{d}}{\pi^{1/4}}\right)\exp\left(-\frac{d^4(ik - i\frac{p'x'}{\hbar})^2}{2d^2}\right)$$

$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right].$$

Which is the required solution □

3.6 Homework Six

3.6.1. Using the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right)S_z = \omega S_z$$

write the Heisenberg equation of motion for the time-dependent operators $S_x(t)$, $S_y(t)$ and $S_z(t)$. Solve them to obtain $S_{x,y,z}$ as functions of time

Solution:
We know the commutaion relation for spin operators $[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$. And since the time derivative of the operator in Heisenberg picture is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{i\hbar}[A, H]$$

We can write the time derivative of the spin operators as

$$\frac{\mathrm{d}}{\mathrm{d}t}S_z = \frac{1}{i\hbar}[S_z, H] = \frac{1}{i\hbar}[S_z, \omega S_z] = \frac{1}{i\hbar}0 = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}S_x = \frac{1}{i\hbar}[S_x, H] = \frac{1}{i\hbar}[S_x, \omega S_z] = -\frac{\omega}{i\hbar}i\hbar S_y = -\omega S_y$$
$$\frac{\mathrm{d}}{\mathrm{d}t}S_y = \frac{1}{i\hbar}[S_y, H] = \frac{1}{i\hbar}[S_y, \omega S_z] = \omega\frac{1}{i\hbar}i\hbar S_x = \omega S_x$$

By similar fashion we can find the second time derivative of the operators as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_z = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}S_z\right) = \frac{\mathrm{d}}{\mathrm{d}t} = 0$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}S_x\right) = \frac{\mathrm{d}}{\mathrm{d}t}(-\omega S_y) = -\omega^2 S_x$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_y = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}S_y\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\omega S_x) = -\omega^2 S_y$$

Since the first time derivative of operator S_z is zero, it is constant over time. For $\frac{\partial^2}{\partial t^2}S_x = -\omega^2 S_x$ forms a Ordinary Second order differential equation in operator S_x. (Assuming derivatives are well defined for operators) We can write the solution as

$$S_x = Ae^{-i\omega t} \qquad\qquad S_y = Be^{-i\omega t}$$

Where A and B are arbitary constant (complex) numbers. □

3.6.2. Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x)$$

By calculating $[[H, x], x]$ prove

$$\sum_{a'} |\langle a''|x|a'\rangle|^2 (E_{a'} - E_{a''}) = \frac{\hbar^2}{2m},$$

where $|a'\rangle$ is an energy eigneket with eigenvalue $E_{a'}$

Solution:

Since x is Hermitian operator and $V(x)$ is pure function of x the commutator of x and $V(x)$ is zero i.e., $[x, V(x)] = 0$. By similar arguments the commutator of p and $\frac{p^2}{2m}$ is zero i.e., $\left[p, \frac{p^2}{2m}\right] = 0$ So we can calculate the commutator

$$[H, x] = \left[\frac{p^2}{2m} + V(x), x\right] = \frac{1}{2m}\left[p^2, x\right] = -i\hbar\frac{p}{m}$$

Also we can simplify the commutator as

$$[[H, x], x] = \left[i\hbar\frac{p}{m}, x\right] = -\frac{i\hbar}{m}[p, x] = -\frac{\hbar^2}{m}$$

the expectation value of the operator $[[H, x], x]$ can be calculated as

$$\begin{aligned}\langle [[H,x],x] \rangle &= \langle a''|[[H,x],x]|a''\rangle \\ &= \langle a''|[Hx - xH, x]|a''\rangle \\ &= \langle a''|Hx^2 - xHx - xHx + x^2H|a''\rangle \\ &= \langle a''|Hx^2|a''\rangle + \langle a''|x^2H|a''\rangle - 2\langle a''|xHx|a''\rangle \\ &= E_{a''}\langle a''|x^2|a''\rangle + E_{a''}\langle a''|x^2|a''\rangle - 2\langle a''|xHx|a''\rangle \\ &= 2E_{a''}\langle a''|x^2|a''\rangle - 2\langle a''|xHx|a''\rangle\end{aligned}$$

Now the quantity $\langle a''|x^2|a''\rangle$ can be written as

$$\langle a''|x^2|a''\rangle = \langle a''|xx|a''\rangle = \sum_{a'} \langle a''|x|a'\rangle \langle a'|x|a''\rangle = \sum_{a'} |\langle a''|x|a'\rangle|^2$$

Similarly we can express $\langle a''|xHx|a''\rangle$ as

$$\langle a''|xHx|a''\rangle = \sum_{a'} \langle a''|xH|a'\rangle \langle a'|x|a''\rangle = \sum_{a'} E_{a'} \langle a''|x|a'\rangle \langle a'|x|a''\rangle = \sum_{a'} E_{a'} |\langle a''|x|a'\rangle|^2$$

Finally these can be substitued to give

$$\langle [[H,x],x] \rangle = 2\sum_{a'}(E_{a'} - E_{a''})|\langle a''|x|a'\rangle|^2$$

But since we calcluated the epectation value to be $-\frac{\hbar^2}{m}$ we can write the expression

$$\sum_{a'}(E_{a'} - E_{a''})|\langle a''|x|a'\rangle|^2 = \frac{\hbar^2}{2m}$$

This is the rquired expression. □

3.7 Homework Seven

3.7.1. Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x)$$

By calculating $[[H, x], x]$ prove

$$\sum_{a'} |\langle a''|x|a'\rangle|^2 (E_{a'} - E_{a''}) = \frac{\hbar^2}{2m},$$

where $|a'\rangle$ is an energy eigneket with eigenvalue $E_{a'}$

Solution:

Since x is Hermitian operator and $V(x)$ is pure function of x the commutator of x and $V(x)$ is zero i.e., $[x, V(x)] = 0$. By similar arguments the commutator of p and $\frac{p^2}{2m}$ is zero i.e., $\left[p, \frac{p^2}{2m}\right] = 0$ So we can calculate the commutator

$$[H, x] = \left[\frac{p^2}{2m} + V(x), x\right] = \frac{1}{2m}\left[p^2, x\right] = -i\hbar\frac{p}{m}$$

Also we can simplify the commutator as

$$[[H,x],x] = \left[i\hbar\frac{p}{m},x\right] = -\frac{i\hbar}{m}[p,x] = -\frac{\hbar^2}{m}$$

the expectation value of the operator $[[H,x],x]$ can be calculated as

$$\begin{aligned}\langle[[H,x],x]\rangle &= \langle a''|[[H,x],x]|a''\rangle \\ &= \langle a''|[Hx-xH,x]|a''\rangle \\ &= \langle a''|Hx^2 - xHx - xHx + x^2H|a''\rangle \\ &= \langle a''|Hx^2|a''\rangle + \langle a''|x^2H|a''\rangle - 2\langle a''|xHx|a''\rangle \\ &= E_{a''}\langle a''|x^2|a''\rangle + E_{a''}\langle a''|x^2|a''\rangle - 2\langle a''|xHx|a''\rangle \\ &= 2E_{a''}\langle a''|x^2|a''\rangle - 2\langle a''|xHx|a''\rangle\end{aligned}$$

Now the quantity $\langle a''|x^2|a''\rangle$ can be written as

$$\langle a''|x^2|a''\rangle = \langle a''|xx|a''\rangle = \sum_{a'}\langle a''|x|a'\rangle\langle a'|x|a''\rangle = \sum_{a'}|\langle a''|x|a'\rangle|^2$$

Similarly we can express $\langle a''|xHx|a''\rangle$ as

$$\langle a''|xHx|a''\rangle = \sum_{a'}\langle a''|xH|a'\rangle\langle a'|x|a''\rangle = \sum_{a'}E_{a'}\langle a''|x|a'\rangle\langle a'|x|a''\rangle = \sum_{a'}E_{a'}|\langle a''|x|a'\rangle|^2$$

Finally these can be substitued to give

$$\langle[[H,x],x]\rangle = 2\sum_{a'}(E_{a'} - E_{a''})|\langle a''|x|a'\rangle|^2$$

But since we calcluated the epectation value to be $-\frac{\hbar^2}{m}$ we can write the expression

$$\sum_{a'}(E_{a'} - E_{a''})|\langle a''|x|a'\rangle|^2 = \frac{\hbar^2}{2m}$$

This is the rquired expression. □

3.7.2. Consider a function, known as the **correlation function**, defined by

$$C(t) = \langle x(t)x(0)\rangle$$

where $x(t)$ is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of a one dimensional harmonic oscillator.

Solution:

The operator in Heisenberg picture change with time. Time evolution of operator x is Heisenberg picture is

$$x(t) = x(0)\cos\omega t + \frac{\sin\omega t}{m\omega}p(0)$$

where $x(0)$ and $p(0)$ are position and momentum operator at time $t=0$. Thus the correlation function becomes

$$\begin{aligned}C(t) = \langle x(t)x(0)\rangle &= \left\langle x(0)^2\cos\omega t + \frac{\sin\omega t}{m\omega}p(0)(x0)\right\rangle \\ &= \left\langle x(0)^2\right\rangle\cos\omega t + \langle p(0)x(0)\rangle\frac{\sin\omega t}{m\omega}\end{aligned}$$

If we denote $x(0)$ and $p(0)$ by just x and p and the ground state of harmonic oscillator by $|0\rangle$ we get

$$C(t) = \langle 0|x^2|0\rangle \cos\omega t + \langle 0|px|0\rangle \frac{\sin\omega t}{m\omega}$$

The operator x and p in terms of creation and annihilation operator are

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$
$$x^2 = \frac{h}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2); \qquad px = i\frac{\hbar}{2}(a^2 + aa^\dagger - a^\dagger a - (a^\dagger)^2)$$

Thus the ground state expectation value of operators become

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2)|0\rangle = \frac{\hbar}{2m\omega}\langle 0|aa^\dagger|0\rangle = \frac{\hbar}{2m\omega}$$
$$\langle 0|px|0\rangle = \frac{i\hbar}{2}\langle 0|a^2 - aa^\dagger + a^\dagger a - (a^\dagger)^2|0\rangle = \frac{i\hbar}{2}\langle 0|aa^\dagger|0\rangle = \frac{i\hbar}{2}$$

Thus the correlation function becomes

$$C(t) = \frac{\hbar}{2m\omega}\cos\omega t + \frac{i\hbar}{2}\frac{\sin\omega t}{m\omega}$$

□

3.7.3. Show that for one-dimensional simple haronic oscillator,

$$\langle 0|e^{kx}|0\rangle = \exp\left[-\frac{k^2\langle 0|x^2|0\rangle}{2}\right]$$

where x is the position *operator*.
Solution:
The expectation value of x^2 in goround state is

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}$$

So the RHS of above expression becomes $\exp\left[-\frac{k^2\hbar}{4m\omega}\right]$ The LHS can be evaluated as

$$\langle 0|e^{ikx}|0\rangle = \int dx' \langle 0|e^{ikx}|x'\rangle\langle x'|0\rangle$$
$$= \int dx' e^{ikx'}\langle 0|x'\rangle\langle x'|0\rangle$$
$$= \int dx' e^{ikx'}|\langle x'|0\rangle|^2$$

The ground state wave function for harmonic oscillator is

$$\langle x'|0\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]$$

Where x_0 is $\sqrt{\frac{\hbar}{m\omega}}$. Thus the expectation value becomes

$$\begin{aligned}
\langle 0|e^{ikx}|0\rangle &= \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\int e^{ikx'}\exp\left[-\left(\frac{x'}{x_0}\right)^2\right]dx' \\
&= \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\int \exp\left[-\left(\frac{x'}{x_0}\right)^2 + ikx'\right]dx' \\
&= \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\int \exp\left[-\left(\frac{x'}{x_0} - \frac{ikx_0}{2}\right)^2 - \frac{k^2x_0^2}{4}\right]dx' \\
&= \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\exp\left(-\frac{k^2x_0^2}{4}\right)\int \exp\left[-\left(\frac{x'}{x_0} - \frac{ikx_0}{2}\right)^2\right]dx' \\
&= \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\exp\left(-\frac{k^2x_0^2}{4}\right)\left(\frac{\pi^{1/4}\sqrt{x_0}}{1}\right) \\
&= \exp\left[-\frac{k^2\hbar}{4m\omega}\right]
\end{aligned}$$

This shows the LHS and RHS are equal proving the proposition. □

On the other hand

$$\begin{aligned}
\langle 0|e^{ikx}|0\rangle &= \langle 0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)}|0\rangle \\
&= \langle 0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}a}e^{ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger}|0\rangle \\
&= \langle 0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}a}e^{ik\sqrt{\frac{\hbar}{2m\omega}}}|1\rangle \\
&= e^{ik\sqrt{\frac{\hbar}{2m\omega}}}\langle 0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}a}|1\rangle \\
&= e^{ik\sqrt{\frac{\hbar}{2m\omega}}}\langle 0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}}|0\rangle \\
&= e^{ik\sqrt{\frac{\hbar}{2m\omega}}}e^{ik\sqrt{\frac{\hbar}{2m\omega}}}\langle 0|0\rangle \\
&= e^{2ik\sqrt{\frac{\hbar}{2m\omega}}}
\end{aligned}$$

I don't see why this approach leads to the wrong solution?

3.7.4. Let

$$J_\pm = \hbar a_\pm^\dagger a_\mp, \qquad J_z = \frac{\hbar 2}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \qquad N = a_+^\dagger a_+ + a_-^\dagger a_-,$$

where $a_\pm$ and $a_\pm^\dagger$ are the annihilation and creation operators of two *independent* simple harmonic oscillators satisfying the usual simple harmonic oscillator commutation relations. Prove

$$[J_z, J_\pm] = \pm\hbar J_\pm, \qquad [\mathbf{J}^2, J_z] = 0, \qquad \mathbf{J}^2 = \left(\frac{\hbar}{2}\right)^2 N\left[\frac{N}{2}+1\right].$$

Solution:

The commutator $[J_z, J_+]$ can be calculated as

$$
\begin{aligned}
[J_z, J_+] &= \frac{\hbar^2}{2}\left[a_+^\dagger a_+ - a_-^\dagger a_-, a_+^\dagger a_-\right] \\
&= \frac{\hbar^2}{2} a_+^\dagger\left[a_+^\dagger a_-, a_+\right] + a_+^\dagger\left[a_-^\dagger a_-, a_-\right] + \left[a_+^\dagger a_-, a_+^\dagger\right] a_+ + \left[a_-^\dagger a_-, a_+^\dagger\right] a_- \\
&= \frac{\hbar^2}{2} a_+^\dagger\left[a_+^\dagger, a_+\right] a_- - a_+^\dagger\left[a_+^\dagger, a_-\right] a_+ + a_+^\dagger\left[a_-^\dagger, a_-\right] a_- \\
&\qquad - \left(a_+^\dagger\right)^2 [a_+, a_-] - a_-^\dagger\left[a_+^\dagger, a_-\right] a_- - \left[a_+^\dagger, a_-^\dagger\right] a_-^2
\end{aligned}
$$

Since $\left\{a_+.a_+^\dagger\right\}$ and $\left\{a_-.a_-^\dagger\right\}$ are independent operators acting on different (independent) systems. The following commutation relation holds

$$
\left[a_+, a_+^\dagger\right] = 1, \quad \left[a_+^\dagger, a_-\right] = 0, \quad \left[a_-^\dagger, a_+\right] = 0, \quad \left[a_-, a_-^\dagger\right] = 1, \quad [a_+, a_-] = 0
$$

Using these in the expression above we get

$$
\begin{aligned}
[J_z, J_+] &= \frac{\hbar^2}{2}\left[a_+^\dagger \cdot 1 \cdot a_- - a_+^\dagger \cdot 0 \cdot a_+ + a_+^\dagger \cdot 1 \cdot a_- - \left(a_+^\dagger\right)^2 \cdot 0 - a_-^\dagger \cdot 0 \cdot a_- - 0 \cdot a_-^2\right] \\
&= \frac{\hbar^2}{2}\left[2a_+^\dagger a_-\right] = \hbar\left[\hbar a_+^\dagger a_-\right] = \hbar J_+
\end{aligned}
$$

Similarly the commutator of $[J_z, J_-]$ can be calculated to be $[J_z, J_-] = -\hbar J_-$

By definition $\mathbf{J}^2 = J_+J_- + J_z^2 + \hbar J_z$(Sakurai 3.5.24). Using this we can write

$$
\begin{aligned}
[\mathbf{J}, J_z] &= \left[J_+J_- + J_z^2 + \hbar J_z, J_z\right] \\
&= [J_+J_-, J_z] + \left[J_z^2, J_z\right] + \hbar[J_z, J_z] \\
&= J_+[J_-, J_z] + [J_+, J_z]J_z + 0 + 0 \\
&= J_+\{\hbar J_-\} + \{-\hbar J_+\}J_- \\
&= \hbar J_+J_- - \hbar J_+J_- \\
&= 0
\end{aligned}
$$

From definition

$$
\mathbf{J}^2 = J_+J_- + J_z^2 + \hbar J_z
$$

$J_+J_- = \hbar^2 a_+^\dagger a_- a_-^\dagger a_+$. Similarly the other terms in the definition are

$$
\begin{aligned}
J_z^2 &= \frac{\hbar^2}{4}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)^2 \\
&= \frac{\hbar^2}{4}\left\{a_+^\dagger a_+ a_+^\dagger a_+ - a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_-\right\}
\end{aligned}
$$

So $\mathbf{J}^2$ becomes

$$
\begin{aligned}
\mathbf{J}^2 &= \frac{\hbar^2}{4}\left\{a_+^\dagger a_+ a_+^\dagger a_+ - a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_-\right\} + \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ + \frac{\hbar^2}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{4} a_+^\dagger a_+\left(a_+^\dagger a_+ + a_-^\dagger a_-\right) + \frac{\hbar^2}{2}\left(a_+^\dagger a_+ + a_-^\dagger a_-\right)\left(1 + \frac{1}{2}a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{2}\left(a_+^\dagger a_+ + a_-^\dagger a_-\right)\left(\frac{a_+^\dagger a_+}{2} + 1 + \frac{a_-^\dagger a_-}{2}\right) \\
&= \frac{\hbar^2}{2} N\left(\frac{N}{2} + 1\right)
\end{aligned}
$$

Which completes the proof. □

3.8 Homework Eight

3.8.1. Consider a One-Dimensional simple harmonic oscillator.

(a) Using

$$\left.\begin{matrix} a \\ a^\dagger \end{matrix}\right\} = \sqrt{\frac{m\omega}{2\hbar}}\left(x \pm \frac{ip}{m\omega}\right), \quad \left.\begin{matrix} a\left|n\right\rangle \\ a^\dagger\left|n\right\rangle \end{matrix}\right\} = \left\{\begin{matrix} \sqrt{n}\left|n-1\right\rangle, \\ \sqrt{n+1}\left|n+1\right\rangle, \end{matrix}\right.$$

evaluate $\langle m|x|n\rangle$, $\langle m|p|n\rangle$, $\langle m|\{x,p\}|n\rangle$, $\langle m|x^2|n\rangle$ and $\langle m|p^2|n\rangle$.

Solution:

Given the definition of a and $a^\dagger$ we can express operator x and p in terms of these operators as

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

The result of operator x and p on any state are

$$x\left|n\right\rangle = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\left|n\right\rangle = \sqrt{\frac{\hbar}{2m\omega}}\big(a\left|n\right\rangle + a^\dagger\left|n\right\rangle\big) = \sqrt{\frac{\hbar}{2m\omega}}\big(\sqrt{n+1}\left|n+1\right\rangle + \sqrt{n}\left|n-1\right\rangle\big)$$
$$p\left|n\right\rangle = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)\left|n\right\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\big(a^\dagger\left|n\right\rangle - a\left|n\right\rangle\big) = i\sqrt{\frac{\hbar m\omega}{2}}\big(\sqrt{n+1}\left|n+1\right\rangle - \sqrt{n}\left|n-1\right\rangle\big)$$

We can calculate these as

$$\langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left\langle m\right|\big(\sqrt{n+1}\left|n+1\right\rangle + \sqrt{n}\left|n-1\right\rangle\big) = \sqrt{\frac{\hbar}{2m\omega}}\big[\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\big]$$
$$\left\langle m\right| p \left|n\right\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left\langle m\right|\big(\sqrt{n+1}\left|n+1\right\rangle - \sqrt{n}\left|n-1\right\rangle\big) = i\sqrt{\frac{\hbar m\omega}{2}}\big[\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\big]$$

So this gives the matrix element of all the operators given. □

3.8.2. Consider again a one-dimensional simple harmonic oscillator. Do the following algebriacally– that is, whout using wave functions.

(a) Construct a linear combination of $\left|0\right\rangle$ and $\left|1\right\rangle$ such that $\langle x\rangle$ is as large as possible.

Solution:

Let the linear combination of $\left|0\right\rangle$ and $\left|1\right\rangle$ be $\left|\alpha\right\rangle = r\left|0\right\rangle + se^{i\delta}\left|1\right\rangle$. Where r and s are real. We can always choose r and s real because the overall phase of the state doesn't matter and δ takes care of the phase difference. The dual correspondence of $\left|\alpha\right\rangle$ is $\left\langle\alpha\right| = \left\langle 0\right| r + \left\langle 1\right| e^{-i\delta}s$. The expectation value of operator x in this state is

$$\begin{aligned}\langle\alpha|x|\alpha\rangle &= (\left\langle 0\right| r + \left\langle 1\right| e^{-i\delta}s)x(r\left|0\right\rangle + se^{i\delta}\left|1\right\rangle) \\ &= r^2\langle 0|x|0\rangle + rse^{i\delta}\langle 0|x|1\rangle + rse^{-i\delta}\langle 1|x|0\rangle + s^2\langle 1|x|1\rangle \\ &= r^2\cdot 0 + rse^{i\delta}\sqrt{\frac{\hbar}{2m\omega}} + rse^{-i\delta}\sqrt{\frac{\hbar}{2m\omega}} + s^2\cdot 0 \\ &= rs\sqrt{\frac{\hbar}{2m\omega}}\underbrace{\left(e^{i\delta} + e^{-i\delta}\right)}_{2\cos\delta} = 2\sqrt{\frac{\hbar}{2m\omega}}rs\cos\delta\end{aligned}$$

The maximum value of this expression is when $\delta = 0$. Also if we want normalized state ket then $s = \sqrt{1-r^2}$. The maximum value of $rs = \max(r\sqrt{1-r^2})$. The maximum value can be obtained by

$$\frac{\mathrm{d}(rs)}{\mathrm{d}r} = \frac{\mathrm{d}(r\sqrt{1-r^2})}{\mathrm{d}r} = \sqrt{1-r^2} - r\frac{2r}{2\sqrt{1-r^2}} = \left(\frac{1-r^2-r^2}{\sqrt{1-r^2}}\right) = 0; \qquad \Rightarrow r = \frac{1}{\sqrt{2}}$$

Substuting this value of r for $s = \sqrt{1 - r^2}$ gives $s = \frac{1}{\sqrt{2}}$. So the linear combination of $|0\rangle$ and $|1\rangle$ that maximizes the expectation of x is

$$|\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \rightarrow \langle\alpha| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

The eigenvalue of operator H in general n stae of simple harmonic oscillator is

$$H|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega\,|n\rangle$$

In the schrodinger picture the time evolution of state $|\alpha\rangle$ is

$$\mathcal{U}(t)|\alpha\rangle = e^{-\frac{iHt}{\hbar}}|\alpha\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{iHt}{\hbar}}|0\rangle + e^{-\frac{iHt}{\hbar}}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(e^{-1/2i\omega t}|0\rangle + e^{-3/2i\omega t}|1\rangle\right)$$

The dual correspondence of this time evolved state is

$$\langle\alpha|\mathcal{U}^\dagger(t) = \frac{1}{\sqrt{2}}\left(e^{1/2i\omega t}\langle 0| + e^{3/2i\omega t}\langle 1|\right)$$

Again the operator x on state $|0\rangle$ and $|1\rangle$ are

$$x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(a|0\rangle + a^\dagger|0\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}}|1\rangle$$
$$x|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(a|1\rangle + a^\dagger|1\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{2}|2\rangle + |0\rangle\right)$$

Thus the expectation vlaue becomes

$$\begin{aligned}\langle\alpha|\mathcal{U}^\dagger(t)x\mathcal{U}(t)|\alpha\rangle &= \frac{1}{\sqrt{2}}\left(e^{1/2i\omega t}\langle 0| + e^{3/2i\omega t}\langle 1|\right)x\frac{1}{\sqrt{2}}\left(e^{-1/2i\omega t}|0\rangle + e^{-3/2i\omega t}|1\rangle\right)\\ &= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(e^{i\omega t} + e^{-i\omega t})\\ &= \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t\end{aligned}$$

Thus the maximum expectation value on schrodinger picture is $\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$. Also in the heisenberg picture the time evolution of operator x is

$$x(t) = x(0)\cos\omega t + p(0)\frac{\sin\omega t}{m\omega}$$

So the expectation value of $x(t)$ is given by

$$\langle x(t)\rangle = \langle x(0)\rangle\cos\omega t + \langle p(0)\rangle\frac{\sin\omega t}{m\omega} = \langle\alpha|x|\alpha\rangle\cos\omega t + \langle\alpha|p(0)|\alpha\rangle\frac{\sin\omega t}{m\omega} = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$

So the expectation value are same in both pictures. To calculate $\langle\Delta x^2\rangle$ we need $\langle x^2\rangle$ and $\langle x\rangle$ since we already know $\langle x\rangle$ we can calculate $\langle x^2\rangle$ as

$$\begin{aligned}\langle\alpha|x^2|\alpha\rangle &= \frac{1}{2}\langle 0|x|0\rangle + \frac{1}{2}e^{-i\omega t}\langle 0|x^2|1\rangle + \frac{1}{2}e^{i\omega t}\langle 1|x^2|0\rangle + \frac{1}{2}\langle 1|x^2|1\rangle\\ &= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar}{2m\omega}}(1 + 0 + 0 + 3) = \frac{\hbar}{m\omega}\end{aligned}$$

Thus

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\cos^2\omega t = \frac{\hbar}{2m\omega}(2-\cos^2\omega t) = \frac{\hbar}{2m\omega}(1+\sin^2\omega t)$$

So the variance in the measurement of position in this satate after time t is $\langle \Delta x^2 \rangle = \frac{\hbar}{2m\omega}(1+\sin^2\omega t)$ □

3.8.3. Consider a particle in one dimension bound to a fixed center by a δ-function potential of the form

$$V(x) = -\nu_0 \delta(x), \qquad (\nu_0 \text{ real and positive}).$$

Find the wave function and the binding energy of the ground state. re there excited bound states?
Solution:
Let the particle has total energy E and mass m. The wave function satisfies the schrodinger equation

$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + E\psi(x) = V(x)\psi(x) \tag{3.10}$$

Since there is a delta function potential $V(x) = -\nu_0\delta(x)$ we can divide the wavefunction as two part function If $x \neq 0$, $V(x) = 0$. Then the Schrodingers equation reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \frac{2mE}{\hbar^2}\psi(x) = 0 \tag{3.11}$$

Thes is a well known second order ordinary differential equation whose solution is in the form

$$\psi(x) = Ae^{\sqrt{\frac{2mE}{\hbar^2}}x} + Be^{-\sqrt{\frac{2mE}{\hbar^2}}x}$$

The requirement that the wavefunction should be normalizable requires that $\lim_{x\to\pm\infty}\psi(x) = 0$. We can evaluate this in two parts.

if $x < 0$ Since the function $e^{-\sqrt{\frac{2mE}{\hbar^2}}x}$ blows up for $x \to -\infty$ it requires that $B = 0$ thus the solution in the region $x < 0$ becomes

$$\psi_-(x) = Ae^{\sqrt{\frac{2mE}{\hbar^2}}x} \tag{3.12}$$

if $x > 0$ Since the function $e^{\sqrt{\frac{2mE}{\hbar^2}}x}$ blows up for $x \to \infty$ it requires that $A = 0$ thus the solution in the region $x < 0$ becomes

$$\psi_+(x) = Be^{-\sqrt{\frac{2mE}{\hbar^2}}x} \tag{3.13}$$

The requirement that the wave function must be continuous everywhere (at $x = 0$) requires that

$$\lim_{x\to 0^-}\psi(x) = \lim_{x\to 0^+}\psi(x), \qquad \Rightarrow A = B$$

Integrating (3.10) from $-\epsilon$ to ϵ and taking the limit $\epsilon \to 0$ we get

$$\int_{-\epsilon}^{\epsilon}\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)dx + \int_{-\epsilon}^{\epsilon}E\psi(x)dx = \int_{-\epsilon}^{\epsilon}\nu_0\delta(x)\psi(x)dx$$

$$\lim_{\epsilon\to 0}\left[\frac{\hbar^2}{2m}\left(\psi'_+(\epsilon) - \psi'_-(-\epsilon)\right)\right] + E\lim_{\epsilon\to 0}\left[\psi_+(\epsilon) - \psi_-(-\epsilon)\right] = -\nu_0\psi(0)$$

By the contunity requirement the middle terms goes to zero because at $x = 0$ $\psi_-(x) = \psi_+(x)$ This simplifies to

$$\frac{\hbar^2}{2m}\left[\psi'_+(0) - \psi'_-(0)\right] = -\nu_0\psi(0)$$

We can differentiate (3.13) ad (3.12) to obtain

$$\frac{\hbar^2}{2m}\left[-A\sqrt{\frac{2mE}{\hbar^2}} - A\sqrt{\frac{2mE}{\hbar^2}}\right] = -\nu_0\psi(0), \qquad \Rightarrow \psi(0) = \frac{A\hbar^2}{m\nu_0}\sqrt{\frac{2mE}{\hbar^2}} = A\sqrt{\frac{2E\hbar^2}{m\nu_0^2}}$$

So the complete solution becomes

$$\psi(x) = \begin{cases} Ae^{\sqrt{\frac{2mE}{\hbar^2}}x} & \text{if } x < 0 \\ A\sqrt{\frac{2E\hbar^2}{m\nu_0^2}} & \text{if } x = 0 \\ Ae^{-\sqrt{\frac{2mE}{\hbar^2}}x} & \text{if } x > 0 \end{cases}$$

The normalization condition can be used to calculate the value of A.

We can use the continuty requirement to evaluate the allowed energy

$$\begin{aligned} \lim_{x\to 0^-}\psi_-(x) = \lim_{x\to 0^+}\psi_+(x) &= \psi(0) \\ \Rightarrow A = A &= A\sqrt{\frac{2E\hbar^2}{m\nu_0^2}} \\ \Rightarrow 1 &= \sqrt{\frac{2E\hbar^2}{m\nu_0^2}} \\ \Rightarrow E &= \frac{m\nu_0^2}{2\hbar^2} \end{aligned}$$

So the ground state binding energy is $\frac{m\nu_0^2}{2\hbar^2}$. Since this energy is the only one that satisfies the scrodinger equation, there is no other bound sate energy. □

Chapter 4

Mathematical Physics II

4.1 Homework One

4.1.1. Use the general definition and properties of Fourier transforms to show the following

(a) If $f(x)$ is periodic with period a then $\tilde{f}(k) = 0$, unless $ka = 2\pi n$ for integer n.
Solution:
We know by definition of fourier transform

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-ikt}dt = \tilde{f}(k) \qquad \text{(Defition)}$$

$$\mathcal{F}(f(t-a)) = \int_{-\infty}^{\infty} f(t-a)e^{-ikt}dt = e^{-ika}\tilde{f}(k) \qquad \text{(Shifting property)}$$

Since the function is periodic $f(t) = f(t-a)$ and hence $\mathcal{F}(f(t)) = \mathcal{F}(f(t-a))$. So,

$$\tilde{f}(k) = e^{-ika}\tilde{f}(k); \quad \Rightarrow (e^{-ika} - 1)\tilde{f}(\omega) = 0;$$

Either $f(k) = 0$ Or $e^{-ika} = 1; \quad \Rightarrow ka = 2\pi n$. Which completes the proof. □

(b) The Fourier transform of $tf(t)$ is $d\tilde{f}(\omega)/d\omega$.
Solution:

$$\frac{d}{d\omega}\left(\tilde{f}(\omega)\right) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{\partial}{\partial\omega}\left(f(t)e^{-i\omega t}\right)dt = \int_{-\infty}^{\infty} itf(t)e^{-i\omega t}dt = -i\mathcal{F}(tf(t))$$

So the fourier transform of $tf(t)$ is $\mathcal{F}(tf(t)) = id\tilde{f}(\omega)/d\omega$. □

(c) The Fourier transform of $f(mt + c)$ is

$$\frac{e^{i\omega c/m}}{m}\tilde{f}\left(\frac{\omega}{n}\right)$$

Solution:
Making a change of variable $mt + c = p; \quad t = \frac{p-c}{m}; \quad dt = \frac{1}{m}dp$ so $e^{-i\omega t} = e^{i\omega c/m}e^{-i\omega/mp}$

$$\mathcal{F}(f(mt+c)) = \int_{-\infty}^{\infty} f(mt+c)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(p)e^{i\omega c/m}e^{i\omega p}\frac{1}{m}dp = \frac{e^{i\omega c/m}}{m}\int_{-\infty}^{\infty} f(p)e^{-i\omega/mp}dp = \frac{e^{i\omega c/m}}{m}\tilde{f}\left(\frac{\omega}{m}\right)$$

So the fourier transform of $f(mt + c)$ is shown as required. □

4.1.2. Find the fourier sine transform$\tilde{f}(\omega)$ of of the function $f(t) = t^{-1/2}$ and by differentiating with respect to ω find the differential equation satisfied by it. Hence show that the the sine transform of this function is the function itself.

Solution:

By definition of sine transform $\tilde{f}(\omega) = \int_0^\infty f(t)\sin(\omega t)dt$ we have for $f(t) = t^{-1/2}$.

$$\frac{d}{d\omega}(\tilde{f}(\omega)) = \frac{d}{d\omega}\int_0^\infty \frac{1}{\sqrt{t}}\sin(\omega t)dt = \int_0^\infty \frac{\partial}{\partial\omega}\left(\frac{1}{\sqrt{t}}\sin(\omega t)\right)dt = \int_0^\infty \sqrt{t}\cos(\omega t)dt$$

Integrating the RHS by parts we get

$$\frac{d}{d\omega}(\tilde{f}(\omega)) = \sqrt{t}\frac{sin(\omega t)}{\omega}\bigg|_0^\infty - \int_0^\infty \frac{1}{2\sqrt{t}}\frac{sin(\omega t)}{\omega} = \frac{1}{\omega}\left[\lim_{t\to\infty}\sqrt{t}\sin(\omega t) - 0\right] - \frac{1}{2\omega}\tilde{f}(\omega)$$

(0 assumed)

So the differential equation satisfied by the sine transform is

$$\frac{d}{d\omega}\left(\tilde{f}(\omega)\right) + \frac{1}{2\omega}\tilde{f}(\omega) = 0$$

This differntial equation can be solved as:

$$\frac{d\tilde{f}(\omega)}{d\omega} = -\frac{1}{2\omega}\tilde{f}(\omega); \qquad \Rightarrow \int \frac{d\tilde{f}(\omega)}{f\tilde{(}\omega)} = \int -\frac{d\omega}{2\omega}; \qquad \Rightarrow \ln\left(\tilde{f}(\omega)\right) = -\frac{1}{2}\ln(\omega) + \ln A; \qquad \Rightarrow \tilde{f}(\omega) = A\omega^{-1/2}$$

But since $f(t) = t^{-1/2}$ the value of $f(\omega) = w^{-1/2}$, so from above expression we get.

$$\tilde{f}(\omega) = Af(\omega)$$

Since we have the sine transform $\tilde{f}(\omega) = Af(\omega)$ the sine transform fo this given function is the function itself. □

4.1.3. Prove the equality

$$\int_0^\infty e^{-2at}\sin^2 at\,dt = \frac{1}{\pi}\int_0^\infty \frac{a^2}{4a^4 + w^4}d\omega$$

Solution:

It can be noticed that the LHS of the given equality is the square integral of function $f(t) = e^{-at}\sin(at)$ from 0 to ∞. Since the lower limit is 0 we can take the fourier transform of this function $u(t)f(t)$ where $u(t)$ is the step function

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} u(t)f(t)e^{-i\omega t}dt = \int_0^\infty e^{-at}\sin(at)e^{-i\omega t}dt = \frac{a}{a^2 + (a + i\omega)^2}$$

The absolute value of the fourier transfom of the function is

$$\left|\tilde{f}(\omega)\right| = \left|\frac{a}{a^2 + (a + i\omega)^2}\right| = \frac{a^2}{\sqrt{4a^4 + w^4}}$$

Now by use of Parseval's theorem we have

$$\int_{-\infty}^{\infty} |u(t)f(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega \qquad \text{(Parseval's theorem)}$$

Substituting $f(t)$ and $\tilde{f}(\omega)$ noting that the function $\tilde{f}(\omega)$ is even

$$\int_0^\infty e^{-2at}\sin^2(at)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{a^2}{\sqrt{4a^4+w^4}}\right)^2 d\omega = \frac{1}{\pi}\int_0^\infty \frac{a^4}{4a^4+w^4}d\omega$$

This completes the proof. □

4.1.4. By writing $f(x)$ as an integral involving the δ-function, $\delta(\xi - x)$ and taking the laplace transform of both sides show that the transform of the solution of the equation

$$\frac{d^4y}{dx^4} - y = f(x)$$

for which y and its first three derivatives vanish at $x = 0$ can be written as

$$\tilde{y}(s) = \int_0^\infty f(\xi)\frac{e^{-s\xi}}{s^4-1}d\xi$$

Solution:
The function $f(x)$ can be written as the integral of delta functions as

$$f(x) = \int_0^\infty \delta(\xi - x)f(\xi)d\xi$$

So the Laplace transform of the function is

$$\tilde{f}(s) = \int_0^\infty \left\{\int_0^\infty \delta(\xi - x)f(\xi)d\xi\right\}e^{-sx}dx = \int_0^\infty \left\{\int_0^\infty \delta(\xi - x)e^{-sx}dx\right\}f(\xi)d\xi = \int_0^\infty e^{-s\xi}f(\xi)d\xi$$

Taking the laplace transform of the given differential equation we get

$$s^4\tilde{y}(s) - \tilde{y}(s) = \tilde{f}(s) = \int_0^\infty e^{-s\xi}f(\xi)d\xi; \qquad \Rightarrow \tilde{y}(s) = \int_0^\infty \frac{e^{-s\xi}}{s^4-1}f(\xi)d\xi$$

Now for the solution this function can be expressed as the product of two functions as

$$\tilde{y}(s) = \underbrace{\frac{1}{s^4-1}}_{\tilde{g}(s)}\underbrace{\int_0^\infty f(\xi)e^{-s\xi}d\xi}_{\tilde{f}(s)}$$

The inverse laplace transform of $\tilde{f}(s)$ is simply $f(x)$ and the fourier transform of $\tilde{g}(s)$ can be obtained as

$$g(s) = \mathcal{L}^{-1}\left(\frac{1}{s^4-1}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right]\right) = \frac{1}{2}[\sinh(x) - \sin(x)]$$

Now the laplace inverse of the product of the function is the convolution of inverses so

$$y(x) = f(x) * g(x) = \int_0^x f(\xi)g(x-\xi)d\xi = \frac{1}{2}\int_0^x f(\xi)[\sinh(x-\xi) - \sin(x-\xi)]d\xi$$

Which completes the proof. □

4.2 Homework Two

4.2.1. Solve the differential equation $y'' - 4y' + y = 0; y(0) = 0; \quad y'(0) = 0$ using the laplace transformation.
Solution:
Let $Y(S)$ be the laplace transformation of $y(x)$. Taking the laplace transformation of given differential equation

$$\mathcal{L}\{y'' - 4y' + y\} = \mathcal{L}\{0\}$$
$$s^2Y(s) - y(0) - sy'(0) - 4sY(s) + 4y(0) + Y(s) = 0$$

Substuting the given initial conditions $y'(0) = 0$; and $y(0) = 0$ gives

$$(s^2 - 4s + 1)Y(s) = 0; \qquad \Rightarrow Y(s) = \frac{1}{s^2 - 4s + 1} = \frac{1}{(S-2)^2 - 3} = \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(S-2)^2 - \sqrt{3}^2}$$

The laplace laplace transform is in the form $\frac{a}{(s-m)^2 - a^2}$ and the laplace inverse of this expression is

$$y(s) = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(S-2)^2 - \sqrt{3}^2}\right\} = \frac{1}{\sqrt{3}} e^{2t} \sinh\left(\sqrt{3}t\right)$$

This is the required solution for the differential equation. □

4.2.2. Using the convolution theorem establish the following result:

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} = \frac{1}{2}\left(t\cos(at) + \frac{1}{a}\sin(at)\right)$$

Solution:
The given expression can be written as

$$\frac{s^2}{(s^2 + a^2)^2} = \frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2}$$

It is easily recognized that each part is the Laplace transform of $cos(at)$. So the inverse Laplace transform by convolution theorem is the convolution of $\cos(at)$ with itself

$$\begin{aligned}
\mathcal{L}^{-1}\{\cdot\} = \cos(at) * \cos(at) &= \int_0^t \cos(ax)\cos(at - ax)dx \\
&= \int_0^t \cos(ax)(\cos(at)\cos(ax) + \sin(at)\sin(ax))dx \\
&= \cos(at)\int_0^t \cos^2(ax)dx + \frac{1}{2}\sin(at)\int_0^t \sin(2x)dx \\
&= \cos(at)\left[\frac{2ax + \sin(2ax)}{4a}\right]_0^t + \frac{1}{2}\sin(at)\left[-\frac{\cos(2ax)}{2a}\right]_0^t \\
&= \frac{1}{2}t\cos(at) + \frac{2}{4a}\sin(at)\cos^2(at) - \frac{1}{2}\sin(at)\cdot\frac{2\cos^2(at)}{2a} + \frac{1}{4a}\sin(at) + \frac{1}{2}\frac{\sin(at)}{2a} \\
&= \frac{1}{2}\left(t\cos(at) + \frac{1}{a}\sin(at)\right)
\end{aligned}$$

This is the required inverse fourier transform for the given expression. □

4.2.3. Show that $\mathcal{L}\{Ci(t)\} = -\frac{1}{2s}\ln\left(1+s^2\right)$ where $C(i) = -\int_t^{\infty}\frac{\cos(u)}{u}du$ (The cosine integral).
Solution:
We know the differential under integral is

$$\frac{d}{dt}\int\limits_{u(t)}^{v(t)} f(t,u)du = f(t,v(t))\frac{dv(t)}{dt} - f(t,u(t))\frac{du(t)}{dt} + \int\limits_{u(t)}^{v(t)}\frac{\partial f(u,t)}{\partial t}du$$

Considering $f(t,u) = \frac{\cos(u)}{u}$, $v(t) = R(\text{as } R- \to \infty)$ and $u(t) = t$ we get

$$\frac{dCi(t)}{dt} = \frac{\cos(R)}{R}\cancelto{0}{\frac{dR}{dt}} - \frac{cos(t)}{t}\frac{dt}{dt} + \lim_{R\to\infty}\int\limits_t^R \cancelto{0}{\frac{\partial}{\partial t}\left(\frac{\cos(u)}{u}\right)}du = -\frac{\cos(t)}{t}$$
$$\Rightarrow \qquad tCi'(t) = \cos(t)$$

Taking the laplace transform of both sides and writing $CI(s) \equiv \mathcal{L}\{Ci(t)\}$ we get

$$\mathcal{L}\{tCi'(t)\} = \mathcal{L}\{\cos(t)\}$$
$$\Rightarrow \qquad -\frac{d}{ds}\mathcal{L}\{Ci'(t)\} = \frac{s}{s^2+1}$$
$$-\frac{d}{ds}\left(sCI(S) - Ci(0)\right) = \frac{s}{s^2+1}$$
$$-\frac{d(sCI(s))}{ds} = \frac{s}{s^2+1}$$

This expression is an ordinary differential equation which can be solved as

$$-\int d(sCI(s)) = \int\frac{ds}{s^2+1}$$
$$\Rightarrow \qquad -sCI(s) = \frac{1}{2}\ln\left(s^2+1\right)$$
$$CI(s) = -\frac{1}{2s}\ln\left(s^2+1\right)$$

This is the required Laplace transform of $Ci(s) \equiv CI(s) = -\frac{1}{2s}\ln\left(s^2+1\right)$. □

4.2.4. By performing the rational fraction decomposition, establish the following results:

(a) $\mathcal{L}^{-1}\left\{\frac{s+1}{s(s^2+1)}\right\} = 1 + \sin(t) - \cos(t)$
Solution:
The partial fraction of

$$\frac{s+1}{s(s^2+1)} = \frac{1}{s} - \frac{s-1}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

Now the inver se laplace transform is

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$= 1 - \cos(t) + \sin(t)$$

Which is the required inverse laplace transform □

(b) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+s+1)}\right\}$
Solution:
The partial fraction of this expression is

$$\frac{s+1}{s^2(s^2+s+1)} = -\frac{1}{s^2+s+1} + \frac{1}{s^2} = \frac{1}{s^2} - \frac{1}{(s+1/2)+1-1/4} = \frac{1}{s^2} - \frac{1}{(s+1/2)+(\sqrt{3}/2)^2}$$

The inverse laplace transform is

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+s+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)+(\sqrt{3}/2)^2}\right\} \\ &= t - \frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\end{aligned}$$

Which is the required inverse transform. □

4.3 Homework Three

4.3.1. A cube made of material whose conductivity is k has its six faces the planes $x = \pm a, y = \pm a$ and $z = \pm a$, and contains no internal heat sources. Verify that the temperature distribution

$$u(x, y, z) = A\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{a}\right)\exp\left(-\frac{2k\pi^2 t}{a^2}\right)$$

obeys the appropirate diffusion equation. Across which faces is there heat flow? WHat is the direction and the rate of heat flow at the point $\left(\frac{3a}{4}, \frac{a}{4}, a\right)$ at time $t = a^2/(\kappa\pi^2)$?
Solution:
Since the expression is the product of sinusoids and exponentials, the derivatives are easy to calculate and are by inspection

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\pi^2}{a^2}u; \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\pi^2}{a^2}u; \qquad \frac{\partial u}{\partial t} = -2\frac{\kappa\pi^2}{a^2}$$

Checking this on the diffusion equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 2\frac{\pi^2}{a^2} = \frac{1}{\kappa} - 2\frac{\pi^2}{a^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

clearly satisfies it, Showing this function obeys the temperature diffucsion equation. The direction of heat flow is given by the gradient of function. At $t = \frac{a^2}{\kappa\pi^2}$; $u = A\cos(x\pi/a)\sin(z\pi/a)e^{-2}$

$$\begin{aligned}\nabla u = \frac{\partial u}{\partial x}\hat{\imath} + \frac{\partial u}{\partial z}\hat{k} &= A\frac{\pi e^{-2}}{a}(-\sin(x\pi/a)\sin(z\pi/a)\hat{\imath} + \cos(x\pi/a)\cos(z\pi/a)\hat{k}) \\ &= A\frac{e^{-2}\pi}{a}(-\sin(\pi/4)\sin(\pi)\hat{\imath} + \cos(\pi/4)\cos(\pi)\hat{k}) \\ &= A\frac{e^{-2}\pi}{a}\left(-\frac{1}{\sqrt{2}}\hat{k}\right)\end{aligned}$$

So the rate of heat flow is $\frac{Ae^{-2}\pi}{a\sqrt{2}}$ in the direction of $-\hat{k}$ □

4.3.2. Schrodinger's equation for an non=reativistic particle ina constant potential region can be taken as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = i\hbar\frac{\partial u}{\partial t}$$

(a) Find a solution, separable in the four independent variables, that can be written in the form of a plane wave

$$\psi(x, y, z, t) = A\exp(i(\boldsymbol{k}.\boldsymbol{r} - \omega t))$$

Using the relationships associated with de Broglie ($\boldsymbol{p} = \hbar\boldsymbol{k}$) and Einstein ($E = \hbar\omega$), show that the separation constants must be such that

$$p_x^2 + p_y^2 + p_z^2 = 2mE$$

Solution:
Lets assume the solution $u(x, y, z, t) = XYZT$ where X is purely function of x only and so on with T being pure function of t. Substuting this product in the given PDE we get

$$-\frac{\hbar^2}{2m}(X''YZT + XY''ZT + XYZ''T) = i\hbar XYZT'$$

Where X'' and so on are total second derivative of their only parameters, x and so on. Dividing thorough by the product $XYZT$ we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = i\hbar\frac{T'}{T}$$

Since we assumed that each X, Y, Z, and T are independent of each other the only way the function of independent variables can be equal is if they are each equal to a constant. Let the constant that each side are equal be E. So we get.

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = i\hbar\frac{T'}{T} = E \quad \text{(Separation Constant)}$$

Solving the ordinary differential equan in t we get

$$\frac{T'}{T} = -i\frac{E}{\hbar}; \qquad \Rightarrow \frac{dT}{T} = -i\frac{E}{\hbar}dt; \quad \Rightarrow \ln(T) = -i\frac{E}{\hbar}t; \qquad \Rightarrow T = T_0e^{-i\omega t}; \text{Where } \omega = \frac{E}{\hbar}$$

Also the LHS must equal same constant so

$$\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = -\frac{2mE}{\hbar^2}$$

The LHS of this expression is sum of three independent functions and the RHS is a constant void of any variables under considerations. The only way that can happen is if each independent function is a constant

$$\frac{X''}{X} = -k_x^2; \qquad \frac{Y''}{Y} = -k_y^2; \qquad \frac{Z''}{Z} = -k_z^2$$

Substuting these back in the differential equation imply that they are related by the expression $-k_x^2 - k_y^2 - k_z^2 = -\frac{2mE}{\hbar^2}$. If we write $p_x = \hbar k_x, p_y = \hbar k_y$, and $p_z = \hbar k_z$. Then we get

$$p_x^2 + p_y^2 + p_z^2 = 2mE \tag{4.1}$$

Each ODE in X, Y and Z are well known Harmonic oscillator differential equations and the solution of each are

$$X = X_0e^{-ik_x x}; \qquad Y = Y_0e^{-ik_y y}; \qquad Z = Z_0e^{-ik_z z} \tag{4.2}$$

Where each of X_0, Y_0 and Z_0 are constants. Combining all these in our final solution we get

$$\begin{aligned} u(x, y, z, t) = XYZT &= X_0e^{-ik_x x} \cdot Y_0e^{-ik_y y} \cdot Z_0e^{-ik_z z} \cdot T_0e^{-i\omega t} \\ &= X_0Y_0Z_0T_0e^{-ik_x x - ik_y y - ik_z z - i\omega t} \end{aligned}$$

If we write $A = X_0Y_0Z_0T_0$, $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_x\hat{\boldsymbol{z}}$ and $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + x\hat{\boldsymbol{z}}$ then the solution takes the form

$$u(x, y, z, t) = Ae^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{4.3}$$

Which is the required solution of the given Schrodinger's equation. □

(b) Obtain a different separable solution describing a particle confined to a box of side a (ψ must vanish at the walls of the box). Show that the energy of the particle can only take quantized values

$$E = \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

where n_x, n_y, n_z are integers.
Solution:
If the solution vanish at the wall of box then each solution given by (4.2) should vanish at the wall. So this iimplies

$$\begin{aligned} 0 &= X_0e^{-ik_xa} & 0 &= Y_0e^{-ik_ya} & 0 &= Z_0e^{-ik_za} \\ \Rightarrow k_xa &= \pi n_x & k_ya &= \pi n_y & \Rightarrow k_za &= \pi n_z \\ \Rightarrow k_x &= \frac{\pi n_x}{a} & k_y &= \frac{\pi n_y}{a} & \Rightarrow k_z &= \frac{\pi n_z}{a} \end{aligned}$$

Substuting these values in (4.1) we get

$$-\left(\frac{n_x\pi}{a}\right)^2 - \left(\frac{n_y\pi}{a}\right)^2 - \left(\frac{n_z\pi}{a}\right)^2 = -\frac{2mE}{\hbar^2}; \qquad \Rightarrow \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right) = E$$

Which is the required solution □

4.3.3. Consider possible solutions of Laplaec's equation inside a circular domain as follows

(a) Find the sollution in plane polar corrdinates ρ, ϕ that takes the value $+1$ for $0 < \phi < \pi$ and the value -1 for $-\pi < \phi < 0$ where $\rho = a$.
Solution:
The general solution for the Laplace's equation in plane polar coordinate system, where the solution is finite at $\rho = 0$ is

$$u(\rho, \phi) = D + \sum_n (C_n\rho^n)(A_n \cos n\phi + B_n \sin n\phi)$$

Since the given boundary condition is an odd function of *phi*, the even function term in the above general solution must vanish so, $D = 0$ and $A_n = 0$. The remaining general solution is

$$u(\rho, \phi) = \sum_n \rho^n(B_n \sin n\phi)$$

Where C_n is absorbed inside of B_n □

(b) For a point (x, y) on or inside the circle $x^2 + y^2 = a^2$, identify the angles α and β defined by

$$\alpha = \text{atan}\left(\frac{y}{a + x}\right); \qquad \text{and} \qquad \beta = \text{atan}\left(\frac{y}{a - x}\right)$$

Show that $u(x, y) = (2/\pi)(\alpha + \beta)$ is a solution of Laplace's equation that satisfies the boundary conditions given in (4.3.3a).

Solution:
Using the trigonometric identity of inverse tangents we get

$$u(x,y) = \frac{2}{\pi}(\alpha+\beta) = \frac{2}{\pi}\left(\text{atan}\left(\frac{y}{a+x}\right) + \text{atan}\left(\frac{y}{a-x}\right)\right) = \frac{2}{\pi}\,\text{atan}\left(\frac{2y}{a^2-x^2-y^2}\right)$$

To verify that $u(x,y)$ satisfies the Laplace's equation we have to show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Calculating this expression

$$\frac{\partial u}{\partial x} = \frac{2y\left(-(a-x)^2+(a+x)^2\right)}{\pi\left(y^2+(a-x)^2\right)\left(y^2+(a+x)^2\right)}; \quad \frac{\partial u}{\partial y} = \frac{2\left((a-x)\left(y^2+(a+x)^2\right)+(a+x)\left(y^2+(a-x)^2\right)\right)}{\pi\left(y^2+(a-x)^2\right)\left(y^2+(a+x)^2\right)}$$

Similarly the second partial derivatives of each is

$$\frac{\partial^2 u}{\partial x^2} = \frac{4ay}{\pi\left(y^2+(a+x)^2\right)^2} + \frac{4ay}{\pi\left(y^2+(a-x)^2\right)^2} + \frac{4xy}{\pi\left(y^2+(a+x)^2\right)^2} - \frac{4xy}{\pi\left(y^2+(a-x)^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{8ay\left(-a^4-2a^2x^2-2a^2y^2+3x^4+2x^2y^2-y^4\right)}{\pi\left(a^2-2ax+x^2+y^2\right)^2\left(a^2+2ax+x^2+y^2\right)^2}$$

On adding $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ we find that it is identically zero. So it satisfies the laplace's equation. On at the boundary $a^2 = x^2+y^2$ and inside the boundary $a^2 > x^2+y^2$ so $a^2 \geq x^2+y^2$. On the boundary

$$u(x,y) = \frac{2}{\pi}\,\text{atan}\left(\frac{2y}{a^2-x^2-y^2}\right) = \frac{2}{\pi}\text{sgn}(2y)\frac{\pi}{2} = \text{sgn}(y)$$

Where $\text{sgn}(x)$ is the sign function. But on boundary $y = a\sin\phi$ where a is the radius and ϕ is the azimuthal angle. The function $\sin\phi$ is positive for $0<\phi<\pi$ and negative for $-\pi<\phi<0$, so

$$u(x,y) = \text{sgn}(y) = \text{sgn}(\sin\phi) = \begin{cases} 1 & 0<\phi<\pi \\ -1 & -\pi<\phi<0 \end{cases}$$

Thus the function satisfies Laplace's equation and also the boundary condition. □

(c) Deduce a Fourier series expansion for the function

$$\text{atan}\left(\frac{\sin\phi}{1+\cos\phi}\right) + \text{atan}\left(\frac{\sin\phi}{1-\cos\phi}\right)$$

Solution:
Again by trigonometric identity

$$f(\phi) = \text{atan}\left(\frac{\sin\phi}{1+\cos\phi}\right) + \text{atan}\left(\frac{\sin\phi}{1-\cos\phi}\right) = \text{atan}\left(\frac{2\sin\phi}{1-\sin^2\phi-\cos^2\phi}\right) = \frac{\pi}{2}\text{sgn}(\sin\phi) = \begin{cases} \pi/2 & 0<\phi<\pi \\ -\pi/2 & -\pi<\phi<0 \end{cases}$$

Let the fourier series of this function $f(\phi)$ be

$$f(\phi) = \frac{a_0}{2} + \sum_n a_n\cos n\phi + b_n\sin n\phi$$

This is a well known periodic square wave function. It is an odd function so $a_n = 0$ whose fourier series is given by

$$a_n = 0;\ \text{and}\ b_n = \frac{\pi}{n}(1-(-1)^n)$$

So the required fouerier series of the function is

$$f(\phi) = \text{atan}\left(\frac{\sin\phi}{1+\cos\phi}\right) + \text{atan}\left(\frac{\sin\phi}{1-\cos\phi}\right) = \sum_{n=1}^{\infty} \frac{\pi}{n}(1-(-1)^n)\sin n\phi$$

This is the required fourier series of the function. □

4.3.4. A conducting spherical shell of radius a cut round its equator and the two halves connected to voltages $+V$ and $-V$. Show that an expression for the potential at the point (r, θ, ϕ) anywhere inside the two hemispheres is

$$u(r,\theta,\phi) = V\sum_{n=0}^{\infty} \frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(n+1)!}\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta)$$

Solution:
For the spherical split spherical shell maintained at two differential potentials, let the potential everywhere inside the spherical shell be v. Since we know electric field is given by $\boldsymbol{E} = \nabla v$ and since for Electric field $\nabla \cdot \boldsymbol{E} = 0$. We get $\nabla \cdot \boldsymbol{E} = \nabla \cdot \nabla v = \nabla^2 v = 0$. So the potential satisfies the Laplace's equation. If we suppose v as a function of r, θ, ϕ in spherical coordinate system, then the solution to Laplace's equation in spherical coordinate system is given by

$$v(r,\theta,\phi) = \sum_{l,m}^{\infty}(Ar^l + Br^{-(l+1)})(C\cos m\phi + D\sin m\phi)(EP_l^m(\cos\theta) + FQ_l^m(\cos\theta))$$

Where $Q_l^m(x)$ and $P_l^m(x)$ are solution to the associated Legendre's equations. And all other constants are determined by boundary condition.

Since we have finite potential at at the center of sphere $r = 0$, the coefficient $B = 0$. Also since we have spherical symmetry and the potential is single valued function $m = 0$. Also we have finite potential at poles of sphere wchich correspond to $\theta = \{0, \pi\}$ and $Q_l^m(1)$ diverges, we have $F = 0$. Also $P_l^0(x) = P_l(x)$ where $P_l(x)$ are legendre polynomials. Owing to these boundary conditions the most general solution is

$$v(r,\theta,\phi) = \sum_l A_l r^l P_l(\cos\theta) \tag{4.4}$$

Since there is no ϕ dependence, let the potential at surface be denoted by v_a which is clearly jut function of θ.

$$v_a(\theta) = v(a,\theta,\phi) = \sum_l A_l a^l P_l(\cos\theta)$$

If we multiply both sides by $P_k(\cos\theta)$ and and integrate with respect to $d(\cos\theta)$ from 0 to 1 using the fact that Legendre's polynomials are orthogonal, $\int P_k P_l = \delta_{kl}$ we get.

$$\begin{aligned}
\int_0^1 v_a(\theta)P_k(\cos\theta)d(cos\theta) &= \int_0^1 \left(\sum_l A_l a^l P_l(\cos\theta)P_k(\cos\theta)d(\cos\theta)\right) \\
&= \sum_l \left(\int_0^1 A_l a^l P_l(\cos\theta)P_k(\cos\theta)d(\cos\theta)\right) \\
&= \sum_l A_l a^l \delta_{lk} = A_k a^k
\end{aligned}$$

So the coefficient A_k is given by

$$A_k = \frac{1}{a^k}\int_0^1 v_a(\theta)P_k(\cos\theta)d(\cos\theta) \tag{4.5}$$

The recurrance relation of Legendre polynomials can be used to evaluate the integrals as

$$(2n+1)P_n = P'_{n+1}(x) - P'_{n-1}(x) \tag{4.6}$$

Integrating (4.6) we get,

$$\int P_n = \frac{1}{2n+1}(P_{n+1}(x) - P_{n-x}(x)) + K$$

Since Potential can have any arbitrary reference we can choose the integration constant to be $K = 0$. Using this fact in (4.5) we get

$$A_k = \frac{1}{a^k(2n+1)} \tag{4.7}$$

As given in the problen on the upper hemisphere the potential is $+V$ and on the lowe hemisphere the potential is $-V$, It can be mathematically represented as

$$v_a(\theta) = \begin{cases} V & \text{if } 0 < \theta < \frac{\pi}{2} \\ -V & \text{if } \frac{\pi}{2} < \theta < \pi \end{cases}$$

Substuting this in (4.5) we get and writing $x = \cos\theta$

$$\begin{aligned} A_k &= \frac{1}{a^k}\int_0^1 VP_k(x)dx \\ &= \frac{V}{a^k}\frac{1}{2k+1}\Big([P_{k+1}(x) - P_{k-1}(x)]_0^1\Big) \\ &= \frac{V}{a^k}\frac{1}{2k+1}(P_{k+1}(1) - P_{k-1}(1) - P_{k+1}(0) + P_{k-1}(0)) \\ &= \frac{V}{a^k}\frac{1}{2k+1}(P_{k-1}(0) - P_{k+1}(0)) \end{aligned}$$

Since

$$P_n(0) = \begin{cases} \dfrac{(-1)^n(2n)!}{2^{2n}n!^2}, & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

For evn value of k, both $k-1$ and $k+1$ are odd and hence $P_{k-1}(0) = 0$ and $P_{k-1}(0) = 0$. Foe even k,

$$A_k = \frac{V}{a^k}\frac{1}{2k+1}(0 - 0 - 1 + 1) = 0$$

But for odd value of k, $k+1$ and $k-1$ are even, hence both $P_{k-1}(1) = P_{k+1}(1) = 1$ and writing $k = 2n+1$

$$\begin{aligned} A_k &= \frac{V}{a^k}\frac{1}{4n+3}\left(\frac{(-1)^{2n}(2(2n)!}{2^{2(2n)}(2n)!^2} - \frac{(-1)^{2(n+1)}(2(2(n+1))!}{2^{2(2(n+1))}(2(n+1))!^2}\right) \\ &= \frac{(4n!)}{den} & = \frac{V}{a^k}\frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(n+1)!} \end{aligned}$$

Using this coefficient in (4.4) we get

$$v(r,\theta,\phi) = V\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(n+1)!}\left(\frac{r}{a}\right)^{2k+1}P_{2n+1}(\cos\theta)$$

Which is the required potential function inside the spherical region. □

4.4 Homework Four

4.4.1. A slice of biological material of thickness L is placed into a solution of a radioactive isotope of constant concentration C_0, at time $t = 0$. For a later time t find the concentration of radioactive ions at a depth x inside one of its surfaces if the diffusion constant is κ.

Solution:

The diffusion equation with diffusion constant κ is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

Using the separation of variable technique for the solution the solution can be written as $u(x,t) = X(x)T(t)$ where X and T are pure functions of x and t respectively. Substituting this solution in the solution we get

$$\frac{X''}{X} = \frac{1}{\kappa}\frac{T'}{T} = -\lambda^2$$

The constant is chosen to be a negative number so that the exponential solution is finite at infinite time. The time part of solution is

$$\frac{T'}{T} = -\kappa\lambda^2; \quad \Rightarrow \int \frac{dT}{T} = \int -\kappa\lambda^2 dt; \quad \Rightarrow \ln T = -\kappa\lambda^2 t + K; \quad \Rightarrow T(t) = De^{-\kappa\lambda^2 t}$$

For the other part $\frac{X''}{X} = -\kappa\lambda^2$ has the solution of the form

$$A\sin\left(\frac{\lambda}{\sqrt{\kappa}}x\right) + B\cos\left(\frac{\lambda}{\sqrt{\kappa}}x\right)$$

The general solution then becomes

$$u(x,t) = \left[A\sin\left(\frac{\lambda}{\sqrt{\kappa}}x\right) + B\cos\left(\frac{\lambda}{\sqrt{\kappa}}x\right)\right]e^{-\lambda^2\kappa t}$$

After sufficient time has passed the concentration throughout the slab should be the concentration of isotopes around it. But the above solution goes to 0 at $t = \infty$. Since adding a constant to the above solution is still the solution to the diffusion equation. We can add a constant to make it satisfy this condition.

Since the concentration is constant at all times on either side of the slab, $u(0,t) = u(L,t) = C_0$ and so $X(0) = X(L) = 0$. So

$$X(0) = Be^{-\lambda^2\kappa t} = 0; \qquad \Rightarrow B = 0$$
$$X(L) = A\sin\left(\frac{\lambda}{\sqrt{\kappa}}L\right) = 0; \qquad \Rightarrow \frac{\lambda}{\sqrt{\kappa}}L = n\pi; \qquad \Rightarrow \lambda = \frac{n\pi\sqrt{\kappa}}{L}$$

Using these two facts we get our general solution to be

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2\kappa}{L^2}t}$$

At $t = 0$ the concentration in the slab must be 0. So $u(x,0) = 0$

$$0 = u(x,0) = C_0 + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right); \qquad \Rightarrow -C_0 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

Again the coefficients A_n can be calculated by using the fact that $\{\sin(nx)\}_n$ form an orthogonal set of function for integer set of n. Integrating the above expression by multiplying by $\sin\left(\frac{m\pi}{L}x\right)$ on both sides gives

$$\begin{aligned}\int_0^L -C_0 \sin\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\ &= \sum_n A_n \frac{1}{L}\delta_{mn} = \frac{A_m}{L} \\ \Rightarrow \quad A_m = L\int_0^L -C_0 \sin\left(\frac{m\pi}{L}x\right)dx &= -LC_0\left\{\frac{2}{m\pi}\frac{1+(-1)^m}{L}\right\}\end{aligned}$$

Using this the general solution becomes

$$u(x,t) = C_0 - \frac{2C_0}{\pi}\sum_m \frac{1+(-1)^m}{m}\sin\left(\frac{m\pi}{L}x\right)e^{-\frac{m^2\pi^2\kappa t}{L^2}}$$

This gives the concentration of radioactive isotope inside the slab at all times. □

4.4.2. Determine the electrostatic potential in an infinite cyinder split lengthwise in four parts and charged as shown.

Solution:

Because the sides of cylindrical are conducting the potential is constant for $u(a,\phi,z)$ where a is the radius of cylinder. It follows that for all z, $u(\rho,\phi)$ is the same. So the potential satisfies plane polar form of laplaces equation which has the general solution

$$u(\rho,\phi) = (C_0 \ln\rho + D_0)\sum_n (A_n\cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n})$$

Since we expect finite solution at $\rho = 0$, $D_n = 0$ otherwise it $\rho^{-n} = \infty$ which won't satisfy boundary condition. By similar arguments $C_n = 0$ Also since at $\rho = a$ the solution is an odd function which causes $D_0 = 0$ and $A_n = 0$. The general solution that is left is

$$u(\rho,\phi) = \sum_n B_n\rho^n \sin n\phi$$

Again the coefficients B_n can be calculated by using the fact that $\{\sin n\phi\}_n$ form an orthogonal set of function for integer set of n. Integrating the above expression by multiplying by $\sin m\phi$ on both sides gives

$$\begin{aligned}\int_0^{2\pi} u(a,\phi)\sin m\phi d\phi &= \int_0^{2\pi}\sum_n A_n a^n \sin n\phi \sin m\phi d\phi \\ &= \sum_n A_n a^n \int_0^{2\pi} \sin n\phi \sin m\phi d\phi \\ &= \sum_n A_n a^n \frac{2\pi}{2}\delta_{mn} = A_m a^m \pi \\ \Rightarrow \quad A_m &= \frac{1}{\pi a^m}\int_0^{2\pi} u(a,\phi)\sin m\phi d\phi\end{aligned}$$

Since in the given problem $u(a, \phi)$ has different values for different ϕ we get

$$\begin{aligned}
A_m &= \frac{1}{\pi a^m}\left\{\int_0^{\pi/2} V\sin m\phi d\phi - \int_{\pi/2}^{\pi} V\sin m\phi d\phi + \int_{-\pi}^{-\pi/2} V\sin m\phi d\phi - \int_{-\pi/2}^{2\pi} V\sin m\phi d\phi\right\}\\
&= \frac{V}{\pi a^m}\left\{-\frac{1}{m}\cos\left(\frac{\pi m}{2}\right) + \frac{1}{m} - \frac{(-1)^m}{m} + \frac{1}{m}\cos\left(\frac{\pi m}{2}\right)\frac{(-1)^m}{m} - \frac{1}{m}\cos\left(\frac{\pi m}{2}\right)\frac{1}{m}\cos\left(\frac{\pi m}{2}\right) - \frac{1}{m}\right\}\\
&= \frac{V}{\pi a^m}\left\{\frac{2(-1)^m}{m} - \frac{4}{m}\cos\left(\frac{\pi m}{2}\right) + \frac{2}{m}\right\}\\
&= \frac{V}{\pi a^m}\left\{\frac{1}{m}\left(-(-1)^{\frac{m}{2}}((-1)^m + 1) - (-1)^m + 1\right)\right\}
\end{aligned}$$

So the final solution becomes

$$u(\rho, \phi) = \frac{V}{m\pi}\left\{1 - (-1)^{\frac{m}{2}}((-1)^m) - (-1)^m + 1\right\}\left(\frac{\rho}{a}\right)^m \sin(m\phi)$$

This gives the potential everywhere inside the cylinder. □

4.4.3. A heat-conducting cylindrical rod of length L is thermally isnulated over its lateral surface and its ends are kept at zero temperature. the initial temperature of the rod is $u(x) = u_0$. using the diffusion equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

and the boundary conditions $u(0, t) = u(L, t) = 0$ and the initial condition $u(x, 0) = u_0$, obtain the solution $u(x, t)$ of the above equation.

Solution:

The general solution to the diffusion equation is

$$u(x, t) = (A\sin(\lambda x) + B\cos(\lambda x))e^{-\lambda^2 a^2 t}$$

Given initial condition $u(0, t) = 0$

$$u(0, t) = e^{-\lambda^2 a^2 t}(B\cos(\lambda x)) = 0$$

Since function has to be 0 at all times the only way this can happen for all t is $B = 0$ Also the other boundary condition is $u(L, t) = 0$ gives

$$u(L, t) = e^{-\lambda^2 a^2 t}A\sin(\lambda L) = 0$$

Since $A = 0$ will give us the trivial solution 0 the only way this function can go to zero at all time is $\sin(\lambda L) = 0$ which implies

$$\sin(\lambda L) = 0; \quad \Rightarrow \lambda L = n\pi; \qquad \Rightarrow \lambda = \frac{n\pi}{L}; \quad (n \geq 1)$$

Since the solution can be linear combination of all n so the solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\lambda^2 a^2 t}\sin\left(\frac{n\pi}{L}x\right)$$

But since the initial condition is that the temperature of the rod is u_0 to begin with. The above solution clearly goes to zero at $t = 0$ and $x = 0$. Adding a constant to a solution of differential equation is still a valid solution, to satisfy this condition we can add a constant u_0. The valid general solution then becomes

$$u(x,t) = u_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda^2 a^2 t} \sin\left(\frac{n\pi}{\lambda} x\right)$$

At $t = 0$ the the solution reduces to

$$u(x,0) = u_0 + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right); \qquad \Rightarrow \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right) = -u_0$$

Sine $\sin(nx)$ forms an orthogonal set of function for integer set of n. We can find A_n by integrating above expression multiplied with $\sin mx$

$$\begin{aligned}
\int_0^l -u_0 \sin\left(\frac{m\pi}{L} x\right) dx &= \int_0^l \sum_n A_n \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{L}\right) dx \\
&= \sum_n A_n \int_0^l \sin\left(\frac{n\pi}{L}\right) \sin\left(\frac{m\pi}{L} x\right) dx \\
&= \sum_n A_n \frac{l}{2} \delta_{nm} = \frac{l}{2} A_m \\
\Rightarrow \qquad A_m &= -\frac{2u_0}{l} \int_0^l \sin\left(\frac{m\pi}{L} x\right) dx = -\frac{2u_0}{l} \left(l \frac{1-(-1)^m}{m} \right)
\end{aligned}$$

Using this in the solution we get the final solution as

$$u(x,t) = u_0 - 2u_0 \sum_{m=1}^{\infty} \left(\frac{1-(-1)^m}{m} \right) \sin\left(\frac{m\pi}{L}\right) e^{-\lambda^2 a^2 t}$$

This gives the temperature as a function of position and time in the given cylindrical body. □

4.4.4. Consider the semi-infinite heat conducting medium defined by the region $x \geq 0$, and arbitrary y and z. Let it be initially at at 0 temperature and let its surface $x = 0$, have prescribed variation of temperature $u(0,t) = f(t)$ for $(t \geq 0)$. Show that the solution of the above diffusion equation can be written as

$$u(x,t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{3/2}} f(\tau) d\tau$$

Solution:
Since the temperature conduction of a material satisfies the diffusion equation, the diffusion equation can be written as.

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Since the parameters of this problems are $t \to \{0, \infty\}$ and $x \to \{0, \infty\}$, we can take the laplace transform of the equation with respect to the variable t which results in

$$\int_0^{\infty} a^2 \frac{\partial^2}{\partial x^2} u(x,t) e^{-st} dt = \int_0^{\infty} \frac{\partial}{\partial t} u(x,t) e^{-st} dt$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_0^{\infty} u(x,t) e^{-st} dt = \frac{1}{a^2} \int_0^{\infty} \frac{\partial}{\partial t} u(x,t) e^{-st} dt$$

Assuming $u(x,t) = g(t)$, the RHS of above expression is the laplace transform of derivative of $g(t)$ which is $sG(t) - g(0)$ which can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}U(x,s) = \frac{1}{a^2}(sU(x,s) - u(x,0))$$

The term $u(x,0)$ is the initial temperature of the material body under construction, since the body is initially at 0 temperature $u(x,0) = 0$, using this and rearranging gives

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}U(x,s) - \frac{s}{a^2}U(x,s) = 0$$

This is a very well known second order Ordinary Differential equation whose solution is of the form

$$U(x,s) = Ae^{-x\sqrt{s}/a} + Be^{x\sqrt{s}/a}$$

But since the material body is infinitely long in $x \geq 0$ the solution is finite at $x = \infty$ which implies that $B = 0$. Also at the near end of the material $x = 0$ the temperature $u(0,t) = f(t)$ is given. The laplace transform of which is $U(0,s) = F(s)$. So

$$U(0,s) = Ae^0; \qquad \Rightarrow A = U(0,s) = F(s)$$

This reduces the solution in the form

$$U(x,s) = F(s)e^{-x\sqrt{s}/a}$$

At this point the solution $u(x,t)$ is the inverse laplace transform of $U(x,s)$. If the expression is taken as product of $F(s)$ and $e^{-x\sqrt{s}/a}$ the solution is the convolution of inverses of these.

Looking at the result we expect, the inverse laplace transfrom must tbe

$$\mathcal{L}^{-1}\left\{e^{-x\sqrt{s}/a}\right\} = \frac{xe^{-\frac{x^2}{4a^2t}}}{2\sqrt{\pi}at^{\frac{3}{2}}}$$

I checked this in sympy and got the following

So the inverse laplace transform of $U(x,s)$ is

$$u(x,t) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}\left\{e^{-x\sqrt{s}/a}\right\} = \int_0^\infty \frac{x}{2\sqrt{\pi}a}\frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{3/2}}f(\tau)d\tau$$

Since the integration is with respect to τ the variable x is consant for integration which leads to

$$u(x,t) = \frac{x}{2a\sqrt{\pi}}\int_0^t \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{3/2}}f(\tau)d\tau$$

Which is the required solution of the heat equation. □

4.5 Homework Five

4.5.1. A string of length l is initially stretched straignt, its ends are fixed for all t. At $t = 0$, its points are given the velocity $v(x) = \left(\frac{\partial y}{\partial t}\right)_{t=0}$ s shown in the diagram. Determine the shape of string at time t, that is, find the displacement as a function of x and t in the form of a series.

In [1]:
```python
import sympy as smp
from sympy.integrals import transforms as strn
smp.init_printing();
```

In [2]:
```python
x,t = smp.symbols('x,t',real=True,positive=True)
s = smp.symbols('s',complex=True)
a,k = smp.symbols('a,kappa',constant=True)
```

In [3]:
```python
#this is the expected laplace transform of the function we need
fx = x/(2*a*smp.sqrt(smp.pi))*smp.exp(-x**2/(4*a**2*t))/t**smp.Rational(3,2) ; fx
```

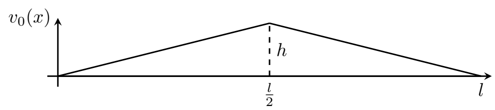
Out[3]:

$$\frac{xe^{-\frac{x^2}{4a^2t}}}{2\sqrt{\pi}at^{\frac{3}{2}}}$$

In [4]:
```python
#calculates the laplace transform of above function
strn.laplace_transform(fx,t,s)
```

Out[4]:

$$\left(e^{-\frac{\sqrt{s}x}{a}},\quad -\infty,\quad 0 < \Re(s) \wedge \left|\text{periodic}_{\text{argument}}\left(\text{polar\_lift}^2(a), \infty\right)\right| < \frac{\pi}{2}\right)$$



Solution:
The motion of the string is guided by the wave equation which can be written as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

If we suppose the solution $y(x,t) = X(x)T(t)$ then substuting these and dividing thourgh by XT we obtain

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

The above solution is composed of two parts, each function of independent variables, the only way they can be equal is if they are equal to constant, let the constant that they are equal be k^2.

$$\frac{X''}{X} = k^2; \qquad \Rightarrow X = A\sin(kx) + B\cos(kx)$$
$$\frac{1}{c^2}\frac{T''}{T} = k^2; \qquad \Rightarrow T = D\sin(kct) + E\cos(kct)$$

So the solution to the differential equation becomes,

$$u(x,t) = [A\sin(kx) + B\cos(kx)][D\cos(kcx) + E\sin(kcx)]$$

But since the string is stationary at both ends. At $x = 0$ and $x = L$

$$0 = B\cos(kx)[D\cos(kct) + E\sin(kct)]$$

The only way it can be zero for all t is if $B = 0$. And also since the string has no displacement to begin with $u(x,0) = 0$. The only way this can happen similarly is if $D = 0$. The solution then becomes

$$u(x,t) = A\sin(kx)\sin(kct)$$

Also since $u(l,t) = 0$ for all t, the only way this can happen is if $k = \frac{n\pi}{l}$. Since we have different possible values of n for solution, the linear combination of all will be the most general solution

$$u(x,t) = \sum_n A_n \sin\Big(\frac{n\pi}{l}x\Big)\sin\Big(\frac{n\pi}{l}ct\Big)$$

The velocity of the string at he begining is

$$u'(x,t) = \sum_n A_n \frac{n\pi c}{l}\sin\Big(\frac{n\pi}{l}x\Big)\cos\Big(\frac{n\pi}{l}ct\Big); \qquad \Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = u'(x,0) = \sum_n \frac{n\pi c}{l}\sin\Big(\frac{n\pi}{l}x\Big)$$

The coefficients A_n can be found by usual "Fourier Trick" as

$$A_n = \frac{2}{n\pi c}\int_0^l u'(x,0)\sin\Big(\frac{n\pi}{L}x\Big)dx$$

Since the given velocity function is two part function we obtain A_n as

$$\begin{aligned}
A_n &= \frac{2}{n\pi c}\left[\int_0^{l/2} u'(x,0)\sin\Big(\frac{n\pi}{l}x\Big)dx + \int_{l/2}^{l} u'(x,0)\sin\Big(\frac{n\pi}{l}x\Big)dx\right] \\
&= \frac{2}{n\pi c}\left[\int_0^{l/2} \frac{2h}{l}x\sin\Big(\frac{n\pi}{L}x\Big)dx + \int_{l/2}^{l} -\frac{2h}{l}(x-l)\sin\Big(\frac{n\pi}{l}x\Big)dx\right] \\
&= \frac{8hl}{\pi^3 cn^3}\sin\left(\frac{\pi n}{2}\right)
\end{aligned}$$

Substiting this back into the solution we have

$$u(x,t) = \sum_{n=0}^{\infty}\frac{8hl}{\pi^3 cn^3}\sin\left(\frac{\pi n}{2}\right)\sin\Big(\frac{n\pi}{l}x\Big)\sin\Big(\frac{n\pi}{l}ct\Big)$$

This gives the position of every point in the string as a function of time. □

4.5.2. Consider the semi-infinite region $y > 0$. For $x > 0$, the surface $y = 0$ is maintained at a temperature $T_0 e^{-x/l}$, fo $x < 0$. The surface $y = 0$ is insulated, so that no head flows out or in. Find the equilibrium temperature at point $(-l, 0)$

Solution:

The general differential equation of the temperature diffusion is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\kappa}\frac{\partial}{\partial t}T$$

. But since at equilibrium the term $\frac{\partial}{\partial t}T = 0$. The general differential equation becomes

$$\frac{\partial^2}{\partial x^2}T(x,y) + \frac{\partial^2}{\partial y^2}T(x,y) = 0 \tag{4.8}$$

The temperature of the system $T(x,y)$ should go to zero as $y \to \infty$. Also since the surface $y = 0$ is insulated for $x < 0$. The heat flow at for $x < 0$ is $\frac{\partial}{\partial t}T(x,y)\big|_{0^-} = 0$. So by contunity of the function at $y = 0$, the rate of change of temperature with y at $y = 0^+$ should equal zero, so $\frac{\partial}{\partial y}T(x,0) = 0$. So the effective boundary condition becomes

$$T(x,0) = f(x) = \begin{cases} T_0 e^{-x/l} & (x > 0) \\ ? & (x < 0) \end{cases} \qquad \frac{\partial}{\partial y}T(x,y)\bigg|_{y=0} = g(x) = \begin{cases} ? & (x > 0) \\ 0 & (x < 0) \end{cases}$$

Taking the fourier transform of (4.8) with respect to the variable x we get

$$\int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}T(x,y)e^{ikx}dx+\int_{-\infty}^{\infty}\frac{\partial^2}{\partial y^2}T(x,y)e^{ikx}dx; \qquad \Rightarrow -k^2\tilde{T}(k,y)+\frac{\mathrm{d}^2}{\mathrm{d}y^2}\tilde{T}(k,y)=0$$

Where $\tilde{T}(K,y)$ is the Fourier transform of $T(x,y)$ in variable x. Since this is a well known differential equation whose solution can be written as

$$\tilde{T}(k,y)=\Phi(k)e^{-ky}$$

Where $\Phi(k)$ is an unknown function to be determined by the boundary conditions. The required solution is the inverse Fourier transform is expression

$$T(x,y)=\mathcal{F}^{-1}(\tilde{T}(k,y))=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}e^{-ky}\Phi(k)dk \tag{4.9}$$

Since we know the various parts at $y=0$ substuting the anove function for $y=0$ gives

$$T(x,0)=f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\Phi(x)dx; \qquad \Rightarrow \Phi(x)=\int_{-\infty}^{\infty}f(x)e^{-ikx}dx \tag{4.10}$$

We will substitue k by $\sqrt{k^2+\lambda^2}$ so that our solution will be in the limit $\lambda\to 0$.. Differentiating the function with respect to y and setting $y=0$ we get

$$\frac{\partial}{\partial y}T(x,y)=g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial y}e^{-\sqrt{k^2+\lambda^2}y}\Phi(k)dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}-\sqrt{k^2+\lambda^2}e^{-\sqrt{k^2+\lambda^2}y}\Phi(k)dx$$

Setting $y=0$ in above expression and taking fourier inverse transform of both sides gives

$$\sqrt{k^2+\lambda^2}\Phi(x)=\int_{-\infty}^{\infty}g(x)e^{-ikx}dx \tag{4.11}$$

We can solve (4.10) and (4.11) with different parts of known $f(x)$ and $g(x)$. From (4.10) we get

$$\Phi(x)=\underbrace{\int_{-\infty}^{0}f(x)}_{\text{Unknown function}}\quad e^{-ikx}dx+\int_{0}^{\infty}f(x)e^{-ikx}dx=\Phi_{-}(x)+\int_{0}^{\infty}T_0e^{-x/l}e^{-ikx}dx \tag{4.12}$$

$$=\Phi_{-}(x)+\frac{T_0}{i}\frac{1}{k-i/l} \tag{4.13}$$

Similarly solving (4.11) we get

$$\sqrt{k^2+\lambda^2}\Phi(x)=\Psi_{+}(x)+0 \tag{4.14}$$

Form (4.12) and (4.14) we get

$$\Psi_{+}(x)=\sqrt{k^2+\lambda^2}\Phi_{-x}+\frac{T_0}{i}\frac{\sqrt{k^2+\lambda^2}}{k-i/l}$$

Dividing by $\sqrt{k-i\lambda}$ on both sides we get

$$\sqrt{k+i\lambda}\Phi_{-}(k)-\frac{\Psi_{+}(k)}{\sqrt{k-i\lambda}}=\frac{T_0}{i}\frac{\sqrt{k+i\lambda}}{k-i/l} \tag{4.15}$$

Thie simplification this expression finally gives

$$T(x,y)=T_0\mathrm{Re}\left[e^{-x-iy}\left(1-\mathrm{erf}\sqrt{-\frac{x+iy}{l}}\right)\right]$$

This is the solution for the temperature everywhere in the rod. At $(-l, 0)$ we get

$$T(-l,0) = T_0 e^l (1 - \text{erf}(1))$$

This gives the temperature at the required point. □

4.5.3. (a) Deduce the relation $P'_{l+1} - P'_{l-1} = (2l+1)P_l$ and show that $\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}$; $(l \geq 1)$

Solution:

The generating polynomial of the Legendre polynomial is

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_n P_n(x)t^n$$

Taking the partial derivative of both sides of the expression with respect to variable x we get

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1-2xt+t^2}}\right) = \frac{\partial}{\partial x}\sum_n P_n(x)t^n \qquad \Rightarrow \qquad \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_n P'_n(x)t^n$$

This can be pulated to get

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P'_n(x)t^n - t\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

$$\sum_{n=0}^{\infty} P'_n(x)t^n - \sum_{n=0}^{\infty} 2xP'_n(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^{n+2} = \sum_{n=0}^{\infty} P_n(x)t^{n+1}$$

Comparing the coefficient of t^{n+1} on both sides we get

$$P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) = P_n(x) \tag{4.16}$$

Again if we differentiate the generating function with respect to t and compare coefficients, we get

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{4.17}$$

If we differentiate the recurrence relation (4.17) we get

$$(2n+1)P_n(x) + (2n+1)xP'_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{4.18}$$

Also if we multiply (4.16) by $(2n+1)$ we get

$$(2n+1)P'_{n+1}(x) - 2(2n+1)xP'_n(x) + (2n+1)P'_{n-1}(x) = (2n+1)P_n(x) \tag{4.19}$$

If we subtract (4.18) from (4.19) we get

$$(2n+1)P_n = P'_{n+1}(x) - P'_{n-1}(x)$$

. This gives the required expression. The integral can be now written as

$$\int_0^1 P_l(x)dx = \int_0^1 \frac{P'_{l+1} - P'_{l-1}(x)}{2l+1}dx = \left[\frac{P_{l+1} - P_{l-1}(x)}{2l+1}\right]_0^1 = \left[\frac{P_{l+1}(1) - P_{l-1}(1) - P_{l+1}(0) + P_{l-1}(0)}{2l+1}\right]$$

Since $P_n(1) = 1$ for every n the expression simplifies to, and since there is $P_{l-1}(1)$ this will be 1 only if $l \geq 1$, which allows us to write,

$$\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}; \quad (l \geq 1)$$

This is te required integral of the Legendre polynomial in the given range. □

(b) Show that $\int_0^l P_l(x)dx = \frac{P_{l-1}(0)}{l+1}; \qquad (l \geq 1).$

Solution:
From (4.5.3a) we can write

$$\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}; \quad (l \geq 1)$$

We can the use (4.17) to evaluate $P_{l+1}(0)$ which gives

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}; \qquad P_{l+1}(0) = -\frac{l}{l+1}P_{l-1}(0)$$

Substuting this back we get

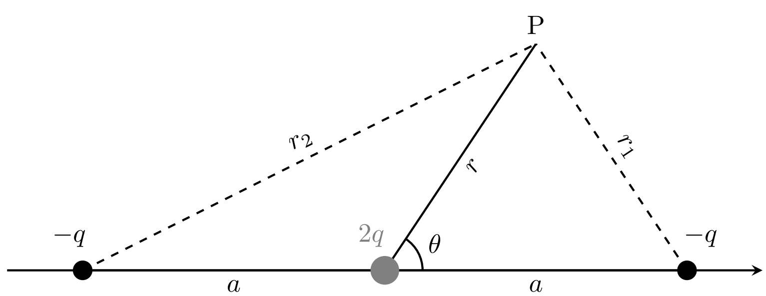
$$\int_0^1 P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1} = \frac{1}{2n+1}\left[P_{l-1}(0) + \frac{l}{l+1}P_{l-1}\right] = \frac{1}{l+1}P_{l-1}(0)$$

Which gives the required expression for the integral. □

4.5.4. A charge $+2q$ is situated at the origin and charges of $-q$ are situated at distances $\pm a$ from it along the polar axis. By relating it to the generating function for the Legendre polynomials, show that the electrostatic potential Φ at a point (r, θ, ϕ) withe $r > a$ is given by

$$\Phi(r, \theta, \phi) = \frac{2q}{4\pi\epsilon r}\sum_{s=1}^{\infty}\left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta).$$

Solution:
Let P be a a general point with coordinate (r, θ) in a particular plane. Since the potential only depends upon r and θ and there is no ϕ dependence, we can cancluate it for a plane polar case, which works for spherical polar as well.



Using the cosine law, the different quantities in the given diagram can be written as

$$r_1^2 = r^2 - 2ra\cos\theta + a^2; \qquad \Rightarrow \quad \left(\frac{r_1}{r}\right)^2 = 1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2$$

Since the generating function of legendre polynomial is

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

If we let $\frac{a}{r} = t$ and $\cos\theta = x$ we get

$$\frac{1}{r_1} = \frac{1}{r}\frac{1}{\sqrt{1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n \tag{4.20}$$

Similarly from r_2 from the diagram

$$r_2^2 = r^2 - 2ra\cos(\pi - \theta) + a^2; \qquad \Rightarrow \quad \left(\frac{r_1}{r}\right)^2 = 1 + 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2$$

Similarly from above expression we get

$$\frac{1}{r_2} = \frac{1}{r}\frac{1}{\sqrt{1 + 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(-\cos\theta)\left(\frac{a}{r}\right)^n \tag{4.21}$$

The potential any point P then becomes

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r} - \frac{q}{4\pi\epsilon_0 r_1} - \frac{q}{4\pi\epsilon_0 r_2} = \frac{q}{4\pi\epsilon_0 r}\left[2 - \frac{1}{r_1} - \frac{1}{r_2}\right]$$

Substuting r_1 and r_2 from (4.20) and (4.21) we get

$$V(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 r}\left[2 - \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty} P_n(-\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

Since $P_n(x) = (-1)^n P_n(-x)$ we get

$$V(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 r}\left[2 - \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty}(-1)^n P_n(\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

Which can be written as

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r}\left[1 - \sum_{n=0}^{\infty}\frac{1 + (-1)^n}{2} P_n(\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

Since $P_0(x) = 1$ for all x we can simplify the expression

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r}\left[\sum_{n=1}^{\infty}\frac{1 + (-1)^n}{2} P_n(\cos\theta)\left(\frac{a}{r}\right)^n\right]$$

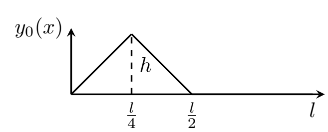
Since the expression $\frac{1+(-1)^n}{2} = 0$ for odd n wec can write

$$V(r,\theta,\phi) = \frac{2q}{4\pi\epsilon_0 r}\sum_{s=1}^{\infty} P_{2s}(\cos\theta)\left(\frac{a}{r}\right)^{2s}$$

Which is the required expression of the potential. □

4.6 Homework Six

4.6.1. A string fixed at both ends and of length l has a zero initial velocity and an initial displacement as shown in the figure. Find the subsquesnt displacement of the string as a function of x and t.



Solution:

The motion of the string is guided by the wave equation which can be written as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

If we suppose the solution $y(x,t) = X(x)T(t)$ then substuting these and dividing thourgh by XT we obtain

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

The above solution is composed of two parts, each function of independent variables, the only way they can be equal is if they are equal to constant, let the constant that they are equal be k^2.

$$\frac{X''}{X} = k^2; \qquad \Rightarrow X = A\sin(kx) + B\cos(kx)$$
$$\frac{1}{c^2}\frac{T''}{T} = k^2; \qquad \Rightarrow T = D\sin(kct) + E\cos(kct)$$

So the solution to the differential equation becomes,

$$u(x,t) = [A\sin(kx) + B\cos(kx)][D\cos(kcx) + E\sin(kcx)] \tag{4.22}$$

But since the string is stationary at both ends. At $x = 0$ and $x = L$

$$0 = B\cos(kx)[D\cos(kct) + E\sin(kct)]$$

The only way it can be zero for all t is if $B = 0$. Substuting $B = 0$ in (4.22) and differentiating with respect to t.

$$\frac{\partial}{\partial t}u(x,t) = A\sin(kx)[kc(-D\sin(kcx) + E\cos(kcx))]; \qquad \Rightarrow u'(x,0) = 0 = A\sin(kx)[Ekc]$$

The only way the above expression can be zero for all x is if $E = 0$. Also since $u(l,t) = 0$ for all t, the only way this can happen is if $k = \frac{n\pi}{l}$. Since we have different possible values of n for solution, the linear combination of all will be the most general solution

$$u(x,t) = \sum_n A_n \sin\Big(\frac{n\pi}{l}x\Big)\cos\Big(\frac{n\pi}{l}ct\Big)$$

The shape of the string at he begining is given as a part function. So the function at $t = 0$ then becomes

$$u(x,0) = \sum_n A_n \sin\Big(\frac{n\pi}{l}x\Big)$$

The coefficients A_n can be found by usual "Fourier Trick" as

$$A_n = \frac{2}{l}\int_0^l u(x,0)\sin\Big(\frac{n\pi}{l}x\Big)dx$$

Since the given velocity function is two part function we obtain A_n as

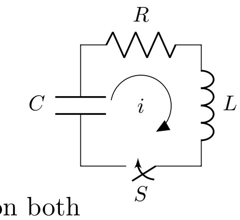
$$\begin{aligned}
A_n &= \frac{2}{l}\left[\int_0^{l/4} u(x,0)\sin\Big(\frac{n\pi}{l}x\Big)dx + \int_{l/4}^{l/2} u'(x,0)\sin\Big(\frac{n\pi}{l}x\Big)dx + \int_{l/2}^{l} u(x,0)\sin\Big(\frac{n\pi}{l}x\Big)dx\right] \\
&= \frac{2}{l}\left[\int_0^{l/4} \frac{2h}{l}x\sin\Big(\frac{n\pi}{L}x\Big)dx + \int_{l/4}^{l/2} -\frac{4h}{l}\left(x - \frac{l}{2}\right)\sin\Big(\frac{n\pi}{l}x\Big)dx\right] \\
&= \frac{8h}{\pi^2 n^2}\left(2\sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{\pi n}{2}\right)\right)
\end{aligned}$$

Substiting this back into the solution we have

$$u(x,t) = \sum_{n=0}^{\infty}\frac{8h}{\pi^2 n^2}\left(2\sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{\pi n}{2}\right)\right)\sin\Big(\frac{n\pi}{l}x\Big)\sin\Big(\frac{n\pi}{l}ct\Big)$$

This gives the position of every point in the string as a function of time. □

4.6.2. A RLC circuit has the charge stored in capacitor q which satisfies the differential equation as



$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0$$

If the charge in the capacitor at time $t = 0$ is $q(0) = q_0$ find the charge as a function of time.

Solution:
Let us assume that the laplace transform of $q(t)$ is $\mathcal{L}\{q(t)\} = Q(s)$. Taking laplace transform on both sides we get

$$L\big\{s^2Q(s) - sq(0) - q'(0)\big\} + R\{sQ(s) - q(0)\} + \frac{1}{C}Q(s) = 0$$
$$\left(Ls^2 + Rs + \frac{1}{C}\right)Q(s) - (Ls + R)q(0) - Lq'(0) = 0$$
$$Q(s) = \frac{L\left(s + q'(0) + \frac{R}{L}\right)}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} = \frac{s}{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} + \frac{q'(0) + \frac{R}{L}}{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

Since there is charge in the capacitor. The initial rate of discharge of capacitor is $q'(0) = \frac{R}{L}$. The denominator can be written as a complete square sum and the expression becomes

$$Q(s) = \frac{s}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} + \frac{q'(0) + \frac{R}{L}}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

Writing $\frac{R}{2L} = \alpha$ and $\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right) = \omega^2$ we get

$$Q(s) = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2} + \frac{\alpha}{\omega}\frac{\omega}{(s + \alpha)^2 + \omega^2}$$

The inverse lapalace transform gives

$$q(t) = e^{-\alpha t}\left(\sin(\omega t) + \frac{\alpha}{\omega}\cos(\omega t)\right)$$

This is the required charge as a function of time in the capacitor. □

4.6.3. Solve the diffusion equation $\frac{\partial^2}{\partial x^2}q(x,t) = \frac{\partial}{\partial t}q(x,t)$ for the initial condition. $q(x,0) = n_0e^{-\alpha|x|}$
Solution:

$$\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial q(x,t)}{\partial t}$$

Taking fourier transform in variable x on both sides

$$\int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}q(x,t) = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}q(x,t); \qquad \Rightarrow -k^2Q(k,t) = \frac{\mathrm{d}}{\mathrm{d}t}Q(k,t)$$

This is a first order differential equation in t whicch has a solution

$$Q(k,t) = A_0e^{-k^2t}$$

Now given the boundary condition $q(x,0) = e^{-\alpha|x|}$ we can calculate the contant A_0 by

$$\begin{aligned}
Q(k,0) = \mathcal{F}(q(x,0)) &= \int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-ikx}dx \\
&= \int_{-\infty}^{0} e^{\alpha x}e^{-ikx}dx + \int_{0}^{\infty} e^{-\alpha x}e^{-ikx}dx \\
&= \int_{-\infty}^{0} e^{(\alpha-ik)x}dx + \int_{0}^{\infty} e^{-(\alpha+ik)x}dx \\
&= \frac{n}{\alpha-ik}e^{\alpha-ik}\Big|_{-\infty}^{0} - \frac{n_0}{x(\alpha+ik)}e^{-x(\alpha-ik)}\Big|_{0}^{\infty} \\
&= \frac{n_0}{\alpha-ik} + \frac{n_0}{\alpha+ik} \\
&= \frac{2\alpha}{\alpha^2+k^2}
\end{aligned}$$

So the solution in k space becomes $Q(k,t) = \frac{2n_0\alpha}{\alpha^2+k^2}$. This then can be used to calculate the solution as

$$q(x,t) = \mathcal{F}^{-1}\left(\frac{2n_0\alpha}{\alpha^2+k^2}e^{-k^2t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2n_0\alpha}{\alpha^2+k^2}e^{ikx}e^{-k^2t}dk = \frac{n_0\alpha}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-k^2t}}{\alpha^2+k^2}dk$$

This gives the gemeral solution. □

4.7 Homework Seven

4.7.1. Show that

$$\frac{1}{|\boldsymbol{r'}-\boldsymbol{r}|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r'^l}{r^{l+1}}\right)\frac{4\pi}{2l+1}Y_l^m(\theta,\phi)^*Y_l^m(\theta',\phi')$$

Solution:
Let the angle between the vectors $\boldsymbol{r'}$ and $\boldsymbol{r}$ be γ. Also let $|\boldsymbol{r}-\boldsymbol{r'}| = r_1$. Then by cosine law wwe have

$$r_1^2 = r'^2 - 2r'r\cos\gamma + r^2;$$

Which can be rearranged to get

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} = \frac{1}{r_1} = \frac{1}{r}\left(\frac{r'}{r} - 2\frac{r'}{r}\cos\gamma + 1\right)^{-1/2} = \frac{1}{r}\sum_{l=0}^{\infty}P_l(\cos\gamma)\left(\frac{r'}{r}\right)^l$$

From the spherical harmonics addition theorem we can write

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_l^m(\theta,\phi)^*Y_l^m(\theta',\phi')$$

Substuting this in the above expression we get

$$\begin{aligned}
\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} &= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_l^m(\theta,\phi)^*Y_l^m(\theta',\phi') \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r'^l}{r^{l+1}}\right)\frac{4\pi}{2l+1}Y_l^m(\theta,\phi)^*Y_l^m(\theta',\phi')
\end{aligned}$$

Clearly this series converses only if $r > r'$ if instead $r' > r$ in the the expression can be rewritten as

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} = \frac{1}{r_1} = \frac{1}{r'}\Big(1 - 2\frac{r}{r'}\cos\gamma + \frac{r}{r'}\Big)^{-1/2} = \frac{1}{r'}\sum_{l=0}^{\infty} P_l(\cos\gamma)\Big(\frac{r}{r'}\Big)^l$$

Using the spherical harmonics addition relation leads to

$$\begin{aligned}\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} &= \frac{1}{r'}\sum_{l=0}^{\infty}\Big(\frac{r}{r'}\Big)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta,\phi)^* Y_l^m(\theta',\phi') \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r^l}{r'^{l+1}}\right)\frac{4\pi}{2l+1} Y_l^m(\theta,\phi)^* Y_l^m(\theta',\phi')\end{aligned}$$

These are the required expressions □

4.7.2. By choosing a suitable form for h in the generating function

$$G(z,h) = \exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)h^n$$

show that the integral representation of the bessel functions of the first kind are given, for integral m by

$$\begin{aligned} J_{2m}(z) &= \frac{(-1)^m}{2\pi}\int_0^{2\pi}\cos(z\cos\theta)\cos(2m\theta)d\theta; & m \geq 1, \\ J_{2m+1}(z) &= \frac{(-1)^m}{2\pi}\int_0^{2\pi}\sin(z\cos\theta)\cos((2m+1)\theta)d\theta & m \geq 0. \end{aligned}$$

Solution:
Let $h = ie^{i\theta}$. With this choice of h we get $h - 1/h = ie^{i\theta} + ie^{-i\theta} = 2i\cos\theta$. This siimplifies the generating function integral to

$$\begin{aligned} e^{iz\cos\theta} &= \sum_{n=-\infty}^{\infty} J_n(z)\big(ie^{i\theta}\big)^n \\ \cos(z\cos\theta) + i\sin(z\cos\theta) &= \sum_{n=-\infty}^{\infty} J_n(z) i^n (\cos\theta + i\sin\theta)^n \\ &= \sum_{n=-\infty}^{\infty} i^n J_n(z)\cos n\theta + i^{n+1} J_n(z)\sin n\theta \end{aligned}$$

Since i^n is real for even n and i^{n+1} is real for odd n. The real part of the expression on RHS is

$$\sum_{m=-\infty}^{\infty} J_{2m} i^{2m}\cos(2m\theta) + J_{2m+1} i^{2m+2}\sin((2m+1)\theta) = \sum_{m=-\infty}^{\infty} J_{2m}(-1)^m\cos(2m\theta) + J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)$$

Thus equating real part on both sides gives

$$\cos(z\cos\theta) = \sum_{m=-\infty}^{\infty} J_{2m}(-1)^m\cos(2m\theta) + J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)$$

Since we know that the set $\{\sin n\theta\}_n$ and $\{\cos n\theta\}_n$ form orthogonal set of functions we can find the expression J_{2m} by usual "Fourier Trick" as

$$\int_0^{2\pi} \cos(z\cos\theta)\cos(2r\theta)d\theta = \int_0^{2\pi}\left(\sum_{m=-\infty}^{\infty} J_{2m}(-1)^m\cos(2r\theta) + J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)\right)\cos 2r\theta d\theta$$
$$= \sum_{m=-\infty}^{\infty}\left(\int_0^{2\pi} J_{2m}(-1)^m\cos(2m\theta)\cos 2r\theta d\theta + \int_0^{2\pi} J_{2m+1}(-1)^{m+1}\sin((2m+1)\theta)\cos 2r\theta d\theta\right)$$
$$= \sum_{m=-\infty}^{\infty}(-1)^m J_{2m}2\pi\delta_{mr} + 0$$
$$= (-1)^r J_{2r}(z)2\pi$$

Rearraning the expression gives since $\frac{1}{(-1)^r} = (-1)^r$

$$J_{2r}(z) = \frac{(-1)^r}{2\pi}\int_0^{2\pi}\cos(z\cos\theta)\cos(2r\theta)d\theta$$

Similarly equating the imaginary part gives

$$\sin(z\cos\theta) = \sum_{m=-\infty}^{\infty} J_{2m}(-1)^m\sin(2m\theta) + J_{2m+1}(-1)^{m+1}\cos((2m+1)\theta)$$

The usual orthogonality gives

$$J_{2r+1}(z) = \frac{(-1)^r}{2\pi}\int_0^{2\pi}\sin(z\cos\theta)\cos((2r+1)\theta)d\theta$$

These concludes the rquirement. □

4.7.3. Find the potential distribution in a hollow conducting cylinder of radius R and length l. The two ends are closed by conducting plates. One end of the plate and the cylindrical wall are held at potential $\Phi = 0$. The other end plate is insulated form the ylindre and held ate potential $\Phi = \phi_0$

Solution:

Since there is no charge source inside the cylinder, the potential in a chargeless region follows the lapalces equation $\nabla^2\Phi = 0$. Using the usual culindrical coordinate system for the problem the general solution of the Laplaces equation in cylindrical solution is given by

$$\Phi(\rho,\phi,z) = [AJ_m(k\rho) + BY_m(k\rho)][C\cos m\phi + D\sin m\phi]\left[Ee^{-kz} + Fe^{kz}\right]$$

Since the potential is finite at $\rho = 0$ at the axis of cylinder, the coefficient $B = 0$ because $Y_m(0) = -\infty$. Since the potential is finite in that region that has to be the case. Also since there is azimuthal symmetry the value of $m = 0$. The general solution then becomes

$$\Phi(\rho,\phi,z) = AJ_0(k\rho)\left[Ee^{-kz} + Fe^{-kz}\right]$$

Since the potential is 0 at $z = 0$ in the bottom end of cylinder. $E + F = 0$; $E = -F$. Absorbing $2F$ into A we get

$$\Phi(\rho,\phi,z) = AJ_0(k\rho)\sinh(kz)$$

Also at the wall of the cylinder $\rho = a$ the potential is zero so

$$0 = \Phi(a,\phi,z) = AJ_0(ka)\sinh(kz)$$

The only way this expression can be zero for all z is if $J_0(ka) = 0$. Which means ka should be the zero of bessel function. Since there are infinite zeros of bessel functions let them be denoted by $\{\alpha_i\}_{i=0}^{\infty}$. This means $ka = \alpha_i; \Rightarrow k_i = \frac{\alpha_i}{a}$ So the general solution becomes

$$\Phi(\rho, \phi, z) = \sum_{i=0}^{\infty} A_i J_0\Big(\frac{\alpha_i}{a}\rho\Big) \sinh\Big(\frac{\alpha_i}{a}z\Big)$$

The coefficient A_i is given by

$$A_i = \frac{2}{J_1^2(\alpha_i) \sinh\big(\frac{\alpha_i}{a}l\big)} \int_0^a \rho\Phi(\rho, \phi, l) J_0\Big(\frac{\alpha_i}{a}\rho\Big) d\rho$$

Since $\Phi(\rho, \phi, l) = \phi_0$ this integral is becomes

$$\begin{aligned} A_i &= \frac{2\phi_0}{J_1^2(\alpha_i) \sinh\big(\frac{\alpha_i}{a}l\big)} \int_0^a \rho J_0\Big(\frac{\alpha_i}{a}\rho\Big) d\rho \\ &= \frac{2\phi_0}{J_1^2(\alpha_i) \sinh\Big(\frac{\alpha_i)}{a}l\Big)} \left[\frac{J_1(\alpha_i)}{\alpha_i}\right] \\ &= \frac{2\phi_0}{\alpha_i J_1(\alpha_i) \sinh\big(\frac{\alpha_i}{a}l\big)} \end{aligned}$$

Substuting this back gives the required general solution

$$\Phi(\rho, \phi, z) = \sum_{i=0}^{\infty} \frac{2\phi_0}{\alpha_i J_1(\alpha_i) \sinh\big(\frac{\alpha_i}{a}l\big)} J_0\Big(\frac{\alpha_i}{a}\rho\Big) \sinh\Big(\frac{\alpha_i}{a}z\Big)$$

This gives the potential everywhere inside the cylinder. □

4.7.4. Show from its definition, that the Bessel function of second kind, and of integer order ν can be written as

$$Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial\mu} - (-1)^\nu \frac{\partial J_{-\mu}(z)}{\partial\mu}\right]_{\mu=\nu}$$

Using the explicit series expression for $J_\mu(z)$, show that $\partial J_\mu(z)/\partial\mu$ can be written as

$$J_\nu(z) \ln\Big(\frac{z}{2}\Big) + g(\nu, z)$$

and deduce that $Y_\nu(z)$ can be expressed as

$$Y_\nu(z) = \frac{2}{\pi} J_\nu(z) \ln\Big(\frac{z}{2}\Big) + h(\nu, z)$$

Where $h(\nu, z)$ lik $g(\nu, z)$, is a power series in z.
Solution:
The definition of the bessel function of second kind is

$$Y_\nu(z) = \lim_{\mu\to\nu} \frac{\cos\mu\pi J_\mu(z) + J_{-\mu}(z)}{\sin\mu\pi}$$

Using L Hopitals rule to evaluate this limit we get

$$Y_\nu(z) = \lim_{\mu\to\mu} \frac{-\pi\sin\mu\pi J_\mu(z) + \cos\mu\pi J_\mu'(z) - (-1)^\mu J_{-\mu}'(z)}{\cos\mu\pi}$$

Since at integer values of ν the value $\cos\nu\pi = 1$ and $\sin\nu\pi = 0$ we get

$$Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial\mu} - (-1)^\nu \frac{\partial J_{-\mu}(z)}{\partial\mu}\right]_{\mu=\nu}$$

For non-integer ν the power series representation of the Bessel function is

$$J_\mu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}$$

Taking derivative with respect to μ we get

$$\begin{aligned}
\frac{\partial J_\mu(z)}{\partial\mu} &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}\ln\left(\frac{z}{2}\right) + \sum_{r=0}^{\infty} -\frac{(-1)^r\Gamma'(r+\mu+1)}{r!\Gamma^2(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r} \\
&= \ln\left(\frac{z}{2}\right)\underbrace{\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}}_{J_\mu(z)} + \underbrace{\sum_{r=0}^{\infty} -\frac{(-1)^r\Gamma'(r+\mu+1)}{r!\Gamma^2(r+\mu+1)}\left(\frac{z}{2}\right)^{\mu+2r}}_{g(\mu,z)} \\
&= \ln\left(\frac{z}{2}\right)J_\mu(z) + g(\mu,z)
\end{aligned}$$

Since $J_{-\mu}(z) = (-1)^\mu J_\mu(z)$. This expression can be reused to calculate the derivative of $J_{-\mu}$. Multiplying both sides of this expression by $(-1)^\mu$ we get

$$\frac{\partial J_{-\mu}(z)}{\partial\mu} = (-1)^\mu \ln\left(\frac{z}{2}\right)J_\mu(z) + (-1)^\mu g(\mu,z)$$

Substuting this back in the expression for the bessel function of second kind we get

$$\begin{aligned}
Y_\nu(z) &= \frac{1}{\pi}\left[\ln\left(\frac{z}{2}\right)J_\mu(z) + g(\mu,z) + (-1)^\mu(-1)^\mu \ln\left(\frac{z}{2}\right)J_\mu(z) + (-1)^\mu g(\mu,z)\right]_{\mu=\nu} \\
&= \frac{1}{\pi}\left[\ln\left(\frac{z}{2}\right)J_\nu(z) + \ln\left(\frac{z}{2}\right)J_\nu(z)\right] + \underbrace{\frac{1}{\pi}[g(\nu,z) + (-1)^\nu g(\nu,z)]}_{h(\nu,z)} \\
&= \frac{2}{\pi}\ln\left(\frac{z}{2}\right)J_\nu(z) + h(\nu,z)
\end{aligned}$$

This gives the requried expression for Bessel function of second kind for integer order. □

4.8 Homework Eight

4.8.1. Consider a damped harmonic oscillator governed by the equation

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f(t)}{m}$$

Where $\lambda^2 - \omega_0^2$ (overdamped case). Suppose the external force $f(t)$ is zero for $t < 0$. Develop the Green's functio and write the solution $x(t)$ satisfying conditions $x(0) = \dot{x}(0) = 0$.

Solution:

Let $G(t,\xi)$ be the solution to the differential equatation with the inhomogenous part replaced by delta function $\delta(t-\xi)$. This can be written as

$$\ddot{G}(t,\xi) + 2\lambda\dot{G}(t,\xi) + \omega_0^2 G(t,\xi) = \delta(t-\xi) \tag{4.23}$$

If $t \neq \xi$ the delta function is zero so the solution for this differential equation for $t < \xi$ is

$$G_-(t,\xi) = e^{-\lambda t}\left(Ae^{t\sqrt{\lambda^2-\omega_0^2}} + Be^{-t\sqrt{\lambda^2-\omega_0^2}}\right) \tag{4.24}$$

Given initial condition $x(0) = 0$. It implies that $G(0,\xi) = 0$, Susbtuting this in the above solution we get

$$G_-(0,\xi) = A + B = 0; \qquad \Rightarrow B = -A$$

Also since $\dot{x}(0) = 0$, we must have $\dot{G}(0,\xi) = 0$. Differentiating (4.24) and substuting $t = 0$ we get

$$\dot{G}_-(t,\xi) = -\lambda\left(Ae^{t\sqrt{\lambda^2-\omega_0^2}} - Ae^{-t\sqrt{\lambda^2-\omega_0^2}}\right)e^{-\lambda t} + \left(A\sqrt{\lambda^2-\omega_0^2}e^{t\sqrt{\lambda^2-\omega_0^2}} + A\sqrt{\lambda^2-\omega_0^2}e^{-t\sqrt{\lambda^2-\omega_0^2}}\right)e^{-\lambda t}$$

$$\Rightarrow \dot{G}_-(0,\xi) = 2A\sqrt{\lambda^2-\omega_0^2} = 0; \qquad \Rightarrow A = 0$$

So the solution for the case $t < \xi$ is trivially $G(t,\xi) = 0$. For $t > \xi$ the solution similarly is

$$G_+(t,\xi) = e^{-\lambda t}\left(Ce^{t\sqrt{\lambda^2-\omega_0^2}} + De^{-t\sqrt{\lambda^2-\omega_0^2}}\right) \tag{4.25}$$

By the contunity requirement of Green's function solution at the vicinity of t ie in $t = \xi \pm \epsilon$

$$\begin{aligned}
\lim_{\epsilon\to 0} G(\xi-\epsilon,\xi) &= \lim_{\epsilon\to 0} G(\xi+\epsilon,\xi) \\
\Rightarrow G_-(t,\xi) = G_+(t,\xi) &= e^{-\lambda\xi}\left(Ce^{\xi\sqrt{\lambda^2-\omega_0^2}} + De^{-\xi\sqrt{\lambda^2-\omega_0^2}}\right) \\
\Rightarrow 0 &= \left(Ce^{\xi\sqrt{\lambda^2-\omega_0^2}} + De^{-\xi\sqrt{\lambda^2-\omega_0^2}}\right)
\end{aligned}$$

$$\Rightarrow C = -De^{-2\xi\sqrt{\lambda^2-\omega_0^2}} \tag{4.26}$$

Also integrating (4.23) in the vicinity of t i.e., for $t = \xi \pm \epsilon$ in the limit $\epsilon \to 0$ we get

$$\lim_{\epsilon\to 0}\left[\dot{G}(t,\xi)\Big|_{t=\xi-\epsilon}^{t=\xi+\epsilon} + \int_{\xi-\epsilon}^{\xi+\epsilon} 2\lambda\dot{G}(t,\xi)dt + \int_{\xi-\epsilon}^{\xi+\epsilon}\omega_0^2 G(t,\xi)dt\right] = \int_{\xi-\epsilon}^{\xi+\epsilon}\delta(t-\xi)dt$$

$$\dot{G}_-(\xi,\xi) - \dot{G}_+(\xi,\xi) + 2\lambda[G_-(\xi,\xi) - G_+(\xi,\xi)] + \frac{\omega_0^2}{2}\left[G_+^2(\xi,\xi) - G_-^2(\xi,\xi)\right] = 1 \tag{4.27}$$

Since the Greens function should be continuous at the vicinity of $t = \xi$, we have $G_+(\xi,\xi) = G_-(\xi,\xi)$ this renders two middle difference in above expression to be zero leaving us with only the difference of derivative. Since $G_-(t,\xi)$ is identically zero its derivative is zero. But the derivative of $G_+(t,\xi)$ at $t = \xi$ is

$$\dot{G}_+(\xi,\xi) = -\lambda\left(Ae^{\xi\sqrt{\lambda^2-\omega_0^2}} + Be^{-\xi\sqrt{\lambda^2-\omega_0^2}}\right)e^{-\lambda\xi} + \left(A\sqrt{\lambda^2-\omega_0^2}e^{\xi\sqrt{\lambda^2-\omega_0^2}} - B\sqrt{\lambda^2-\omega_0^2}e^{-\xi\sqrt{\lambda^2-\omega_0^2}}\right)e^{-\lambda\xi}$$

Substuting C from (4.26) we get

$$\dot{G}_+(\xi,\xi) = 2D\sqrt{\lambda^2-\omega_0^2}e^{-\lambda\xi}e^{\xi\sqrt{\lambda^2-\omega_0^2}}$$

Substuting $\dot{G}_-(\xi,\xi) = 0$ and $\dot{G}_+(\xi,\xi)$ from above in (4.27) we get

$$2D\sqrt{\lambda^2-\omega_0^2}e^{-\lambda\xi}e^{\xi\sqrt{\lambda^2-\omega_0^2}} = 1; \qquad \Rightarrow D = \frac{1}{\sqrt{\lambda^2-\omega_0^2}}e^{-\xi(\lambda-\sqrt{\lambda^2-\omega_0^2})}$$

Using this in (4.26) we get

$$C = \frac{1}{\sqrt{\lambda^2 - \omega_0^2}} e^{-\xi(\lambda + \sqrt{\lambda^2 - \omega_0^2})}$$

Thus $G_+(t, \xi)$ becomes

$$G_+(t, \xi) = \frac{e^{(t-\xi)\lambda}}{\sqrt{\lambda^2 - \omega_0^2}} \left(e^{(t-\xi)\sqrt{\lambda^2 - \omega_0^2}} - e^{-(t-\xi)\sqrt{\lambda^2 - \omega_0^2}} \right)$$

So the rquired Green's function is

$$G(t, \xi) = \begin{cases} 0 & \text{if } t \leq \xi \\ \frac{e^{(t-\xi)\lambda}}{\sqrt{\lambda^2 - \omega_0^2}} \left(e^{(t-\xi)\sqrt{\lambda^2 - \omega_0^2}} - e^{-(t-\xi)\sqrt{\lambda^2 - \omega_0^2}} \right) & \text{if } t > \xi \end{cases}$$

So the soution of the differential equation becomes

$$x(t) = \int_\xi^t G(t, \xi) \frac{f(t)}{m} dt$$

This is the required solution of the differential equation. □

4.8.2. We are to solve $y'' - k^2 y = f(x)(0 \leq x \leq L)$ subject to the boundary conditions $y(0) = y(L) = 0$.

(a) Find Green's function by direct construction.
Solution:

$$\frac{d^2 y}{dx^2} - k^2 y = f(x)$$

for $0 \leq x \leq l$, with $y(0) = y(l) = 0$.
The green's function solution to non homogenous differential equation $\mathcal{L}y = f(x)$ is a solution to homogenous part of the differential equation with the source part replaced as delta function $\mathcal{L}y = \delta(x - \xi)$. The ontained solution is $G(x, \xi)$, i.e., $\mathcal{L}G(x, \xi) = \delta(x - \xi)$. This solution corresponds to the homogenous part only as it is independent of any source term $f(x)$.

$$\frac{d^2}{dx^2} G(x, \xi) - k^2 y = \delta(x - \xi); \qquad \text{with } G(0, \xi) = 0; \text{ and } G(l, \xi) = 0 \text{ for all } 0 \leq \xi \leq l \tag{4.28}$$

Since delta function $\delta(x - \xi)$ is zero everywhere except $x = \xi$ we can find solution for two regions $x < \xi$ and $x > \xi$. For $x < \xi$ let the solution to $\mathcal{L}y = 0$ be $y_1(x)$ and for $x > \xi$ be $y_2(x)$ then

$$y_1''(x) - k^2 y_1(x) = 0; \text{ for } x < \xi; \qquad y_2''(x) - k^2 y_2(x) = 0; \text{ for } x > \xi$$

These are well known second order ODES whose solution are

$$y_1(x) = A \sinh kx + B \cosh kx; \qquad y_2(x) = C \sinh kx + D \cosh kx$$

By the boundary condition $y_1(0) = 0$ and $y_2(l) = 0$. These immediately imply that $B = 0$. Also since the soution to the differential equation must be continuous $y_1(\xi) = y_2(\xi)$. Integrating Eq. (4.28) in the vicinity of ξ we get

$$y'(x) \Bigg|_{\xi_-}^{\xi_+} - k^2 \cancelto{0 \text{ By contunity}}{\int y dx \Bigg|_{\xi_-}^{\xi_+}} = \int_{\xi_-}^{\xi_+} \delta(x - \xi) dx; \Rightarrow y'(\xi_+) - y'(\xi_-) = 1$$

From three different conditions, (i) contunity at ξ, (ii) $y_2(l) = 0$ and (iii) $y_1'(\xi) - y_2'(\xi) = 1$ we get following three linear equations. Using there parameters we get.

$$\begin{aligned} Ck\cosh k\xi + Dk\sinh k\xi - Ak\sinh k\xi &= 1 \\ C\sinh k\xi + D\cosh k\xi - A\sinh k\xi &= 0 \\ C\sinh kl + D\cosh kl &= 0 \end{aligned}$$

Which can be written in the matrix form and solved as.

$$\begin{bmatrix} k\cosh(k\xi) & k\sinh(k\xi) & -k\cosh(k\xi) \\ \sinh(k\xi) & \cosh(k\xi) & -\sinh(k\xi) \\ \sinh(kl) & \cosh(kl) & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ A \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C \\ D \\ A \end{bmatrix} = \begin{bmatrix} \frac{\sinh(k\xi)}{k\tanh(kl)} \\ -\frac{1}{k}\sinh(k\xi) \\ \frac{1}{k}\left(\frac{\sinh(k\xi)}{\tanh(kl)} - \cosh(k\xi)\right) \end{bmatrix}$$

Giving

$$C = \frac{\sinh(k\xi)}{k\tanh(kl)}; \qquad D = -\frac{1}{k}\sinh(k\xi); \qquad A = \frac{1}{k}\left(\frac{\sinh(k\xi)}{\tanh(kl)} - \cosh(k\xi)\right)$$

So the required function is

$$G(x,\xi) = \begin{cases} y_1(x) = \frac{1}{k}\left(\frac{\sinh(k\xi)}{\tanh(kl)} - \cosh(k\xi)\right)\sinh kx = -\frac{\sinh kx \sinh k(L-\xi)}{k\sinh kl} & \text{if } x < \xi \\ y_2(x) = \frac{\sinh(kx)\sinh(k\xi)}{k\tanh(kl)} - \frac{1}{k}\sinh(k\xi)\cosh(kx) = \frac{\sinh k\xi \sinh k(L-x)}{k\sinh kl} & \text{if } x > \xi \end{cases} \tag{4.29}$$

Eq.(4.29) gives the Green's function whcin can be used to find the solution to the differential equation

$$y(x) = \int G(x,\xi)f(\xi)d\xi$$

The solution to the original inhomogenous differential equation can is given by the above expression in terms of Green's function. □

(b) Solave the equation $G'' - k^2G = \delta(x-\xi)$ by the Fourier sine series method. Can you show that the series obtained for $G(x,\xi)$ is equivalent to the solution found under (a).
Solution:
Let sine series of solution G be

$$G = \sum A_n \sin\left(\frac{n\pi}{L}x\right); \qquad G'' = \sum A_n - \frac{n^2\pi^2}{L^2}\sin\left(\frac{n\pi}{L}x\right)$$

Substuting these back in the differential equation we get

$$\sum -A_n\left(\frac{n\pi}{L} - k^2\right)\frac{n\pi}{L}\sin\left(\frac{n\pi}{L}x\right) = \delta(x-\xi)$$

Again let the fourier sine series of delta function be

$$\delta(x,\xi) = \sum B_n \sin\left(\frac{n\pi}{L}x\right)$$

The coefficient B_n can be found as

$$\begin{aligned} \int \delta(x-\xi)\sin\left(\frac{m\pi}{L}x\right)dx &= \int \sum B_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\ \sin\left(\frac{m\pi}{L}\xi\right) &= \sum_n B_n \frac{L}{2}\delta_{mn} = B_m\frac{L}{2} \\ \Rightarrow B_m = \frac{2}{L}\sin\left(\frac{m\pi}{L}\xi\right) \end{aligned}$$

Substuting this back into the differential equation we get

$$\sum -A_n\Big(\frac{n\pi}{L}-k^2\Big)\frac{n\pi}{L}\sin\Big(\frac{n\pi}{L}x\Big)=\delta(x-\xi)=\sum_n\frac{2}{L}\sin\Big(\frac{m\pi}{L}\xi\Big)\sin\Big(\frac{n\pi}{L}x\Big)$$

Comparing the coefficients we get

$$A_n(k^2-\frac{n\pi}{L})\frac{n\pi}{L}=\frac{2}{L}\sin\Big(\frac{n\pi}{L}\xi\Big);\Rightarrow A_n=\frac{2}{(k^2-\frac{n\pi}{L})n\pi}\sin\Big(\frac{n\pi}{L}\xi\Big)$$

Thus the Green's function becomes

$$G=\sum_n A_n\sin\Big(\frac{n\pi}{L}x\Big)=\sum_n\frac{2}{(k^2-\frac{n\pi}{L})n\pi}\sin\Big(\frac{n\pi}{L}\xi\Big)\sin\Big(\frac{n\pi}{L}x\Big)$$

Which is the required sine series of differential equation. The fourier series of solution in part (a) is exactly this. □

4.8.3. Show that the Green's function designed to solve the DE

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\left(k^2x-\frac{m^2}{x}\right)y=f(x) \tag{4.30}$$

subject to $y(0)<\infty$ and $y(a)=0$ reads

$$G(x,\xi)=\begin{cases}\frac{\pi}{2}\frac{[J_m(k\xi)Y_m(ka)-J_m(ka)Y_m(k\xi)]}{J_m(ka)}J_m(kx) & (x\le\xi)\\ \frac{\pi}{2}\frac{[J_m(kx)Y_m(ka)-J_m(ka)Y_m(kx)]}{J_m(ka)}J_m(k\xi) & (x\ge\xi)\end{cases}$$

Also consider the case $J_m(ka)=0$. Show that if $k\neq 0$, then $G(x,\xi)$ does not exist, but $k=0$ $G(x,\xi)$ does exist, although the above form is not applicable. Evaluate $G(x,\xi)$ in this case.

Solution:

Let $G(x,\xi)$ be the solution of the differential equation with the inhomogenous function by a delta function $\delta(x-\xi)$. If $x\neq\xi$ then the delta function is zero and the differential equation is a bessel differential equation. So the solution of bessel differential equation of order m are is

$$G(x,\xi)=AJ_m(kx)+BY_m(kx).$$

We can divide the solution into two parts with $x<\xi$ and $x>\xi$.

Given the boundary condition the solution is finite for $x=0$, the coefficient of bessel function of second kind $Y_m(kx)$ is zero because it blows up at $x=0$.

$$G_-(x,\xi)=y_1(x)=AJ_m(kx);\qquad (x<\xi)$$

For $x>\xi$, the solution $y_2(x)$ can is

$$G_-(x,\xi)=y_2(x)=CJ_m(kx)+DY_m(kx);\qquad (x\ge\xi)$$

Since the given initial condition is $y_2(a)=0$ we get

$$D=-C\frac{J_m(ka)}{Y_m(ka)}$$

So the kernel of solution is begin $y_2(x)=J_m(kx)+qY_m(kx)$ where $q=-\frac{J_m(ka)}{Y_m(ka)}$.

The greens function solution is

$$G(x,\xi) = \begin{cases} \frac{y_1(x)y_2(\xi)}{p(\xi)\mathcal{W}(\xi)} & (x \le \xi) \\ \frac{y_1(\xi)y_2(x)}{p(\xi)\mathcal{W}(\xi)} & (x \ge \xi) \end{cases} \tag{4.31}$$

Where $\mathcal{W}(\xi) = [y_1(x)y_2'(x) - y_1'(x)y_2(x)]_{x=\xi}$ is the wronskian of two independent solution to two parts in the range $[a,b]$ divided by the point $\xi \in [a,b]$

Now we have to calculate the wronskian $\mathcal{W}(y_1(x), y_2(x))$ which can be evaluated as

$$\begin{aligned} \mathcal{W}(y_1(x), y_2(x)) &= \mathcal{W}(J_m(kx), J_m(kx) + qY_m(kx)) \\ &= \mathcal{W}(J_m(kx), J_m(kx)) + q\mathcal{W}(J_m(kx), Y_m(kx)) \\ &= 0 + q\frac{2}{\pi x} \end{aligned}$$

Thus $\mathcal{W}(\xi) = \frac{2q}{\pi\xi}$. We have our $p(\xi) = \xi$. Thus,

$$\begin{aligned} \frac{y_1(x)y_2(\xi)}{p(\xi)\mathcal{W}(\xi)} &= \frac{J_m(kx)[J_m(k\xi) + qY_m(k\xi)]}{\xi \cdot \frac{2q}{\pi\xi}} = J_m(kx)\frac{\pi}{2}\frac{[J_m(k\xi)Y_m(ka) - J_m(ka)Y_m(k\xi)]}{J_m(ka)} \\ \frac{y_2(x)y_1(\xi)}{p(\xi)\mathcal{W}(\xi)} &= \frac{[J_m(kx) + qY_m(kx)]J_m(k\xi)}{\xi \cdot \frac{2q}{\pi\xi}} = \frac{\pi}{2}\frac{[J_m(kx)Y_m(ka) - J_m(ka)Y_m(kx)]}{J_m(ka)}J_m(k\xi) \end{aligned}$$

Using this in green function solution in (4.31) we obtain.

$$G(x,\xi) = \begin{cases} \frac{\pi}{2}\frac{[J_m(k\xi)Y_m(ka) - J_m(ka)Y_m(k\xi)]}{J_m(ka)}J_m(kx) & (x \le \xi) \\ \frac{\pi}{2}\frac{[J_m(kx)Y_m(ka) - J_m(ka)Y_m(kx)]}{J_m(ka)}J_m(k\xi) & (x \ge \xi) \end{cases}$$

Which is the required Green's function solution. □

4.8.4. Consider th eboundary-value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x) \qquad y(0) = 0 \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad (0 \le x \le L).$$

(a) Find the normalized eigenfunctions of the oeprator for teh given boybdary conditions
Solution:
Let $g(x)$ be the eigen function of the operator with $-\lambda^2$ as eigen value thus

$$\frac{\mathrm{d}^2 g(x)}{\mathrm{d}x^2} = -\lambda^2 g(x)$$

We know the solution of this differential equation are

$$g(x) = \sin(\lambda x); \quad g'(x) = \lambda\cos(\lambda x); \quad g'(L) = 0 = \lambda\cos(\lambda L); \qquad \Rightarrow \lambda L = (2n+1)\frac{\pi}{2}$$

Thus the eigenfunctions become

$$g(x) = \sin\left(\frac{(2n+1)\pi}{2L}x\right); \qquad -\lambda^2 = \left(\frac{(2n+1)\pi}{2L}\right)^2$$

This is is the eigenvalue and eigenfunction of the given operator subject to given boundary condition □

(b) Write bilinear formula for Green's function

Solution:

The bilinear form is

$$G(x,\xi) = \sum_n \frac{g_n(x)g_n^*(\xi)}{\lambda_n} = \sum_n \frac{\sin\left(\frac{(2n+1)\pi}{2L}x\right)\sin\left(\frac{(2n+1)\pi}{2L}\xi\right)}{\left(\frac{(2n+1)\pi}{2L}\right)^2}$$

This is the required bilinear form □

Chapter 5

The Standard Model

5.1 Homework One

5.1.1. **(SMIN 1.1)** Consider a vector field in three-dimensional cartesian space:

$$u^i = \begin{pmatrix} xy \\ x^2+2 \\ 3 \end{pmatrix}$$

(a) Compute the components of $\partial_j u^i$.
(b) Compute $\partial_i u^i$
(c) Compute $\partial_j \partial^j u^i$.

Solution:
Since for three-dimensional cartesian space the indices run from 1 through 3,

$$\partial_j u^i = \begin{pmatrix} \partial_1 u^1 & \partial_1 u^2 & \partial_1 u^3 \\ \partial_2 u^1 & \partial_2 u^2 & \partial_2 u^3 \\ \partial_3 u^1 & \partial_3 u^2 & \partial_3 u^3 \end{pmatrix} = \begin{pmatrix} y & 2x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\partial_i u^i = \partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3 = y + 0 + 0 = y$$

Since for cartesian space the metric elements are

$$\partial_i = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}, \qquad g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \partial^j = g^{ij}\partial_i = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\Rightarrow \partial_j \partial^j = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \nabla^2 \qquad \Rightarrow \qquad \partial_j \partial^j u^i = \nabla^2 \begin{pmatrix} xy \\ x^2+2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Each of the components are thus calculated. □

5.1.2. **(SMIN 1.4)** Express the following quantities in naural units, in the form (# GeV)^n.

(a) The current energy density of universe : $\sim 1 \times 10^{-26} kg/m^3$.
(b) 1 angstrom
(c) 1 nanosecond

(d) 1 gigaparsec $\simeq 3 \times 10^{25} m$
(e) The luminosity of the sun $\simeq 4 \times 10^{26} W$

Solution:

$$\frac{1kg}{m^3} = \left(\frac{c^2 J}{\left(\frac{1}{c}s\right)^3}\right) = \left(\frac{c^2 J}{\left(\frac{1}{c}\frac{1}{\hbar}J^{-1}\right)^3}\right) = c^5\hbar^3 J^4 = c^5\hbar^3\left(\frac{1}{e\times 10^9}GeV\right)^4 = \left(\frac{c^{5/4}\hbar^{3/4}}{e\times 10^9}GeV\right)^4$$
$$\frac{1\times 10^{-26}kg}{m^3} = \left(\frac{1\times 10^{-26/4}c^{5/4}\hbar^{3/4}}{e\times 10^9}GeV\right)^4 = (1.01\times 10^{-11}GeV)^4$$

$$1m = \frac{1}{c}s = \frac{1}{c\hbar}J^{-1} = (c\hbar J)^{-1} = \left(\frac{c\hbar}{e\times 10^9}GeV\right)^{-1}$$
$$1\,\text{Å} = 1\times 10^{-10}m = \left(\frac{1\times 10^{10}c\hbar}{e\times 10^9}GeV\right)^{-1} = (1.24\times 10^{-5}GeV)^{-1}$$
$$1Gpc \simeq 3\times 10^{25}m = \left(\frac{1\times 10^{-25}c\hbar}{3\cdot e\times 10^9}GeV\right)^{-1} = (4.13\times 10^{-41}GeV)^{-1}$$

$$1s = \frac{1}{\hbar}J^{-1} = \left(\frac{\hbar}{e\times 10^9}GeV\right)^{-1} \quad \Rightarrow \quad 1ns = 1\times 10^{-9} = \left(\frac{1\times 10^9\hbar}{e\times 10^9}GeV\right)^{-1} = (4.41\times 10^{-15}GeV)^{-1}$$

$$1W = \frac{1J}{1s} = \frac{1J}{\frac{1}{\hbar}J^{-1}} = \hbar J^2 = \left(\frac{\hbar^{1/2}}{e\times 10^9}GeV\right)^2$$
$$L_\odot \simeq 4\times 10^{26}W = \left(\frac{(4\times 10^{26})^{1/2}\hbar^{1/2}}{e\times 10^9}GeV\right)^2 = (3.21\times 10^6 GeV)^2$$

□

5.1.3. **(SMIN 1.6)** Consider a 4-vector

$$A^\mu = \begin{pmatrix}2\\3\\0\\0\end{pmatrix}$$

(a) Compute $A \cdot A = A^\mu A_\mu$.
Solution:
For minkowski space the metric is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \qquad \Rightarrow A_\mu = g_{\mu\nu}A^\nu = \begin{pmatrix}2 & -3 & 0 & 0 & 0\end{pmatrix}$$
$$A\cdot A = A^\mu A_\mu = 2\cdot 2 + (3)\cdot(-3) = -5$$

Thus the dot product is -5.

□

(b) What are the components of $A^{\bar{\mu}}$ if you rotate the coordinate frame around the z-axis through an angle $\theta = {}^{\pi}/_{3}$?
Solution:
The transformation matrix for rotation around z-axis at an angle $\theta = \frac{\pi}{3}$ is

$$\Lambda^{\bar{\mu}}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos {}^{\pi}/_{3} & \sin {}^{\pi}/_{3} & 0 \\ 0 & -\sin {}^{\pi}/_{3} & \cos {}^{\pi}/_{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & {}^{1}/_{2} & {}^{\sqrt{3}}/_{2} & 0 \\ 0 & -{}^{\sqrt{3}}/_{2} & {}^{1}/_{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\nu} A^{\nu} = \begin{pmatrix} 2 \\ 3/2 \\ -3\sqrt{3}/2 \\ 0 \end{pmatrix}$$

These are the required transformed components. □

(c) For your answer in part (5.1.3b), verify that $A^{\bar{\mu}} A_{\bar{\mu}}$ is the same as in part (5.1.3a).
Solution:

$$g^{\bar{\mu}\bar{\nu}} = \Lambda^{\bar{\mu}}_{\mu} \Lambda^{\bar{\nu}}_{\nu} g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad g_{\bar{\mu}\bar{\nu}} = g^{\bar{\mu}\bar{\nu}} \qquad A_{\bar{\mu}} = g_{\bar{\mu}\bar{\nu}} A^{\bar{\nu}} = \begin{pmatrix} 2 \\ -3/2 \\ 3\sqrt{3}/2 \\ 0 \end{pmatrix}$$

$$A_{\bar{\mu}} A^{\bar{\mu}} = 2 \cdot 2 + {}^{3}/_{2} \cdot (-{}^{3}/_{2}) + \left(-3{}^{\sqrt{3}}/_{2}\right) + 3{}^{\sqrt{3}}/_{2} = 4 - \frac{9}{4} - \frac{27}{4} = -5$$

The inner product is -5 as required. □

(d) What are the components $A^{\bar{\mu}}$ if you boost the frame (from (5.1.3a)) a speed $v = 0.6$ in x-direction?
Solution:
For $v = 0.6 = 3/5$ the "gamma factor" is $\gamma = 1/\sqrt{1 - .6^2} = 1.25 = 5/4$ and thus $v\gamma = 0.75 = 3/4$
The transformation matrix under this boost is

$$\Lambda^{\bar{\mu}}_{\mu} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5/4 & 3/4 & 0 & 0 \\ 3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\nu} A^{\nu} = \begin{pmatrix} 19/4 \\ 21/4 \\ 0 \\ 0 \end{pmatrix}$$

which are the required components under boost. □

(e) For your answer in part (5.1.3d), verify that $A^{\bar{\mu}} A_{\bar{\mu}}$ is the same as in part (5.1.3a).
Solution:
Since the Minkowski metric is invariant under boost, the transformed metric is $g_{\bar{\mu}\bar{\nu}} = g_{\mu\nu}$

$$g_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_{\bar{\mu}} = g_{\bar{\mu}\bar{\nu}} A^{\bar{\nu}} = \begin{pmatrix} 19/4 \\ -21/4 \\ 0 \\ 0 \end{pmatrix}, \quad A^{\bar{\mu}} A_{\bar{\mu}} = {}^{19}/_{4} \cdot {}^{19}/_{4} + {}^{21}/_{4} \cdot (-{}^{21}/_{4}) = -5$$

The inner product is -5 as in (5.1.3a). □

5.1.4. **(SMIN 1.10)** Consider a scalar field

$$\phi(x) = 2t^2 - 3x^2$$

(a) Compute the components of $\partial_\mu \phi$.
(b) Compute the components of $\partial^\mu \phi$.
(c) Compute $\partial_\mu \partial^\mu \phi$. (This operation is the **d'Alembertian operator**).

Solution:

$$\partial_\mu \phi(x) = \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \phi(x) = \begin{pmatrix} 2t \\ -6x \\ 0 \\ 0 \end{pmatrix} \qquad \partial^\mu = g^{\mu\nu}\partial_\nu = \begin{pmatrix} \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix}, \quad \Rightarrow \partial^\mu \phi(x) = \begin{pmatrix} 2t \\ 6x \\ 0 \\ 0 \end{pmatrix}$$

The operator

$$\partial_\mu \partial^\mu = \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix} = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \equiv \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\partial_\mu \partial^\mu \phi(x) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi(x) = 4 + 6 = 10$$

Thus the d'Alembertian operator on the given scalar function $\phi(x)$ is 10. □

5.1.5. **(SMIN 1.13)** An excited hydrogen atom emits a $10.2eV$ Lyman$-\alpha$ photon.

(a) What is the momentum of the photon? (Express in natural units.)
(b) As Newton's third law remains in force, what is the kinetic energy of the recoiling ground state hydrogen atom?
(c) What is the recoil speed of proton.

Solution:
For a photon $E = p$ so the momentum of the photon is $10.2eV$.

If Newton's law remain ins force, then the recoiling ground state atom has the same momentum as the outgoing photon. So the recoiling momentum of ground state atom is $10.2eV$.

Mass of proton is $9.38 \times 10^8 eV$, since the mass of electron is negligible coompared to proton let us assume the proton carries all the momentum so,

$$p = \gamma m v \quad \Rightarrow \quad \frac{v}{\sqrt{1-v^2}} = \frac{p}{m} \quad \Rightarrow \quad v = \frac{1}{\sqrt{1 + \left(\frac{m}{p}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{9.38\times 10^8}{10.2}\right)^2}} = 1.08 \times 10^{-8} \equiv 3.26 m/s$$

So the recoil speed of proton is $1.08 \times 10^{-8} \equiv 3.26 m/s$. □

5.2 Homework Two

5.2.1. **(SMIN 2.3)** Consider two particles of equal mass m connected by a spring of constant k and confined tomove in one dimension. The entire ssytem moves without friction. At equilibrium the spring has length L.

(a) Write down the Lagrangian of this system as a function of x_1 and x_2 and their derivatives. Assume $x_2 > x_1$.
Solution:
The kinetic energy of each mass is $\frac{1}{2}m\dot{x_1}^2$ and for the second mass is $\frac{1}{2}m\dot{x_2}^2$. The total compression

in the spring is $x_2 - x_1 - L$ so the total potential energy in the spring is $V = \frac{1}{2}k\left(x_2 - x_1 - L\right)^2$. So the lagrangian of the system becomes

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k\left(x_2 - x_1 - L\right)^2$$

This is the required Lagrngian. □

(b) Write the Euler-Lagrange equation for this system.
Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_1}\right) &= \frac{\partial\mathcal{L}}{\partial x_1} &&\Rightarrow m\ddot{x}_1 = k\left(x_2 - x_1 - L\right)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_2}\right) &= \frac{\partial\mathcal{L}}{\partial x_2} &&\Rightarrow m\ddot{x}_2 = -k\left(x_2 - x_1 - L\right)\end{aligned}$$

These are the required Euler-Lagrange equation of the system. □

(c) Make the change of variables

$$\Delta \equiv x_2 - x_1 - L \qquad X = \frac{1}{2}\left(x_2 + x_2\right)$$

Write the Lagrangian in these new variables.
Solution:
Eliminating x_1 and x_2 between the two transformation variables Δ and X we get

$$\begin{aligned}&x_2 - x_1 = \Delta + L \qquad x_1 + x_1 = 2X\\ &2x_2 = 2X + \Delta + L \qquad \Rightarrow x_2 = X + \frac{1}{2}\left(\Delta + L\right) \qquad \dot{x_2} = \dot{X} + \frac{1}{2}\dot{\Delta}\\ &2x_1 = 2X - \Delta - L \qquad \Rightarrow x_2 = X - \frac{1}{2}\left(\Delta + L\right) \qquad \dot{x_1} = \dot{X} - \frac{1}{2}\dot{\Delta}\end{aligned}$$

Using these variables the lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\left(\dot{X} + \frac{1}{2}\dot{\Delta}\right)^2 + \frac{1}{2}m\left(\dot{X} - \frac{1}{2}\dot{\Delta}\right)^2 + \frac{1}{2}k\Delta^2\\ &= \frac{1}{2}m\left(2\dot{X}^2 + \dot{\Delta}^2\right) + \frac{1}{2}k\Delta^2\end{aligned}$$

This is the lagrangian in the transformed coordinate system. □

5.2.2. **(SMIN 2.7)** Show that the complex Lagrangian $\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*$ is algebraically identical to

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2$$

if $m_1 = m_2 1 = m$ and

$$\phi = \left(\frac{\phi_1 + \phi_2}{2}\right) + i\left(\frac{\phi_1 - \phi_2}{2}\right)$$

Solution:

Assuming the scalar fields ϕ_1 and ϕ_2 are real valued function. The complex field and its conjugate are

$$\phi = \left(\frac{\phi_1+\phi_2}{2}\right) + i\left(\frac{\phi_1-\phi_2}{2}\right) \qquad \phi^* = \left(\frac{\phi_1+\phi_2}{2}\right) - i\left(\frac{\phi_1-\phi_2}{2}\right)$$

$$\begin{aligned}\Rightarrow \phi\phi^* &= \frac{1}{4}\left[(\phi_1+\phi_2)^2 + (\phi_1-\phi_2)^2\right] = \frac{1}{4}\left(\phi_1^2+\phi_2^2+2\phi_1\phi_2+\phi_1^2+\phi_2^2-2\phi_1\phi_2\right)\\ &= \frac{1}{2}(\phi_1^2+\phi_2^2)\end{aligned}$$

$$\begin{aligned}\partial_\mu\phi\partial^\mu\phi^* &= \partial_\mu\left(\frac{\phi_1+\phi_2}{2}+i\frac{\phi_1+\phi_2}{2}\right)\partial^\mu\left(\frac{\phi_1+\phi_2}{2}-i\frac{\phi_1+\phi_2}{2}\right)\\ &= \frac{1}{2}\left(\partial_\mu\phi_1+\partial_\mu\phi_2+i\partial_\mu\phi_1+i\partial_\mu\phi_2\right)\frac{1}{2}\left(\partial^\mu\phi_1+\partial^\mu\phi_2-i\partial^\mu\phi_1-i\partial^\mu\phi_2\right)\\ &= \frac{1}{4}(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_1\partial^\mu\phi_2-i\partial_\mu\phi_1\partial^\mu\phi_1-i\partial_\mu\phi_1\partial^\mu\phi_2\\ &\quad+\partial_\mu\phi_2\partial_\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2-i\partial_\mu\phi_2\partial^\mu\phi_1-i\partial_\mu\phi_2\partial^\mu\phi_2\\ &\quad+i\partial_\mu\phi_1\partial^\mu\phi_1+i\partial_\mu\phi_1\partial^\mu\phi_2+\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_1\partial^\mu\phi_2\\ &\quad+i\partial_\mu\phi_2\partial^\mu\phi_1+i\partial_\mu\phi_2\partial^\mu\phi_2+\partial_\mu\phi_2\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2)\\ &= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2\right)\end{aligned}$$

Substuting these back in to the complex Lagrngian we get

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*\\ &= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2\right) - m^2\frac{1}{2}(\phi_1^2+\phi_2^2)\\ &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2\end{aligned}$$

This shows the two Lagrangian are equivalent. □

5.2.3. **(SMIN 2.9)** Consider a lagrangian of real-valued scalar field:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}c_3\phi^3.$$

(a) Is this Lagrangian Lorentz invariant? It is invariant under C, P, and T transformations individually?
Solution:
Since every term in the lagrangian is a scalar it is trivially Lorentz invariant. As ϕ is real valued scalar its complex conjugate is itself $\phi^* = \phi$ since the $\hat{\mathbf{C}}$ transformation transforms ϕ to ϕ^* which are identical so the Lagrngian is invariant under $\hat{\mathbf{C}}$ transformation.
It is not invariant under $\hat{\mathbf{P}}$ and $\hat{\mathbf{T}}$ transformation. □

(b) What is the dinemsion of c_3?
Solution:
Since the dimension of Lagrangian density is $[E]^4$ and the dimensionality fo ϕ is $[E]$ the dimensionality of c_3 is $[E]^3$ □

(c) What is Euler-Lagrange equation for field?
Solution:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$\partial_\mu\left(\partial^\mu\phi\right) = -m^2\phi - \frac{c_3}{2}\phi^2$$

This is the required Euler-Lagrange equation for the given Lagrangian density. □

(d) Ignoring the c_3 contribution, a free-field solution may be written

$$\phi_0(x) = Ae^{-ip\cdot x} + A^* e^{ip\cdot x}$$

for a complex coefficient A. Consider a lowest-order contribution for $\phi_1 \ll A$ to a peturbation such that $\phi(x) = \phi_0 + \phi_1$. Derive a dynamical eqution for ϕ_1.
Solution:

$$\partial_\mu \partial^\mu (\phi(x)) + m^2 \phi(x) + \frac{c_3}{2}\phi^2 = 0$$

Substuting $\phi = \phi_0 + \phi_1$

$$\Rightarrow \qquad \partial_\mu \partial^\mu (\phi_0(x) + \phi_1(x)) + m^2 (\phi_0(x) + \phi_1(x)) + \frac{1}{2}c_3(\phi_0 + \phi_1)^2 = 0$$
$$\Rightarrow \qquad \partial_\mu \partial^\mu \phi_1(x) + m^2 \phi_1(x) + \frac{1}{2}c_3\phi_0^2 \left(1 + \frac{\phi_1}{\phi_0}\right)^2 = -\partial_\mu \partial^\mu \phi_0 - m^2\phi_0$$
$$\Rightarrow \qquad \partial_\mu \partial^\mu \phi_1(x) + m^2 \phi_1(x) + \frac{1}{2}c_3\phi_0^2 \left(1 + 2\frac{\phi_1}{\phi_0}\right) = -\partial_\mu \partial^\mu \phi_0 - m^2\phi_0$$

The first term in RHS of above expression is

$$\begin{aligned}
\partial_\mu \partial^\mu \phi_0 &= g^{\mu\nu} \partial_\nu (\partial_\mu \phi_0) \\
&= g^{\mu\nu} \partial_\nu \left((-ip_\mu)Ae^{-ip\cdot x} + (ip_\mu)A^* e^{ip\cdot x}\right) \\
&= \partial_\nu \left((-ip^\nu)Ae^{-ip\cdot x} + (ip^\nu)A^* e^{ip\cdot x}\right) \qquad (\text{ Distributing } g^{\mu\nu}) \\
&= (-p^\nu p_\nu)Ae^{-ip\cdot x} + (-p^\nu p_\nu)A^* e^{ip\cdot x} \\
&= -(E^2 - |\boldsymbol{p}|^2)\phi_0
\end{aligned}$$

□

5.3 Homework Three

5.3.1. **(SMIN 3.3)** A particle of mass m and charge q in an electromagnetic field has a Lagrangian

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - q(\phi - \dot{\boldsymbol{r}} \cdot \boldsymbol{A}),$$

where ϕ is the scalar potential, and $\boldsymbol{A}$ is the vector potential.

(a) Suppose (just for the moment) that the potential fields are not explicit functions of x. Use Noether's theorem to compute the conserved quantity of the electromagnetic Lagrangian.
Solution:
Writing the Lagrangian in cartesian coordinate system we get

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\left(\phi(y,z) - \dot{x}A_x - \dot{y}A_y - \dot{z}A_z\right)$$

Since the lagrangian is invariant under translation in $x \to x + \epsilon$ the conserved quantity is

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{\mathrm{d}x}{\mathrm{d}\epsilon} = \qquad m\dot{x} + qA_x$$

So the conserved quantity if the potential fields are independent of x is $m\dot{x} + qA_x$. □

(b) More generally assume that the potential fields vary in space and time. What are the Euler-Lagrange equations for this Lagrangian corresponding to particle position x^i?
Solution:
If the potential fields depend upon space and time the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^i\dot{x}^i\right) - q\left(\phi(x^i) - \dot{x}^i A_i\right)$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^i}\right) &= \frac{\partial\mathcal{L}}{\partial x^i} \\ \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}^i + qA_i\right) &= -q\left(\frac{\partial\phi}{\partial x^i} - \dot{x}^j\frac{\partial A_j}{\partial x^i}\right) \\ \Rightarrow \qquad m\ddot{x}^i + q\frac{\mathrm{d}A_i}{\mathrm{d}t} &= -q\left(\frac{\partial\phi}{\partial x^i} - \dot{x}^j\frac{\partial A_j}{\partial x^i}\right) \end{aligned}$$

These are the required Euler-Lagrange equations. □

(c) Solve the previous solution explicitly for $m\ddot{x}$. Express your final answer as a combination of $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.
Solution:
Specifically for $x^i = x$ the above expression becomes

$$\begin{aligned} m\ddot{x} + q\frac{\mathrm{d}A_x}{\mathrm{d}t} &= -q\left(\frac{\partial\phi}{\partial x} - \dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_y}{\partial x} - \dot{z}\frac{\partial A_z}{\partial x}\right) \\ m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial A_x}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial A_x}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right) &= -q\left(-E_x - \dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_y}{\partial x} - \dot{z}\frac{\partial A_z}{\partial x}\right) \\ m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right) &= qE_x + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}\right) \\ m\ddot{x} &= qE_x + q\left(\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right) \\ m\ddot{x} &= qE_x + q\left(\dot{y}B_z - \dot{z}B_y\right) \\ m\ddot{x} &= qE_x + q\left(\boldsymbol{r}\times\boldsymbol{B}\right)_x \end{aligned}$$

This is the required equation of motion for the x coordinate under given Lagrangian. □

5.3.2. **(SMIN 3.6)** We've seen that a real valued scalar field may be expanded as a plane-wave solution:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]$$

Computer the total anisotropic stress$\int d^3x T^{ij}$ where $i \neq j$, for a real-valued field by integrating over the stress-energy tensor.
Solution:
The stress-energy tensor is

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}$$

For $i \neq j$ $g^{ij} = 0$ so the Tensor reduces to

$$T^{ij} = \frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\partial^j\phi = \partial_i\phi\partial^j\phi$$

For a complex scalar field given we have

$$\partial_i\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}\partial_i\left(e^{-ip\cdot x}\right) + c^*_{\boldsymbol{p}}\left(\partial_i e^{ip\cdot x}\right)\right] = i\int \frac{d^3p}{(2\pi)^3}\frac{p^i}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]$$

And similarly the

$$\partial^j\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}\partial^j\left(e^{-ip\cdot x}\right) + c^*_{\boldsymbol{p}}\left(\partial^j e^{ip\cdot x}\right)\right] = -i\int \frac{d^3p}{(2\pi)^3}\frac{p_j}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]$$

Thus the product is

$$\begin{aligned}
T^{ij} &= i\int \frac{d^3p}{(2\pi)^3}\frac{p^i}{\sqrt{2E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]\cdot -i\int \frac{d^3p}{(2\pi)^3}\frac{p_j}{\sqrt{2E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p^i}{\sqrt{2E_p}}\frac{q_j}{\sqrt{2E_q}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]\left[c_{\boldsymbol{q}}e^{-iq\cdot x} + c^*_{\boldsymbol{q}}e^{iq\cdot x}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p^iq_j}{\sqrt{4E_pE_q}}\left[c_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{-i(p+q)\cdot x} + c_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{-i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{i(p+q)\cdot x}\right]
\end{aligned}$$

Integrating this quantity over the volume yields

$$\int T^{ij}d^3x = \int d^3x\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p^iq_j}{\sqrt{4E_pE_q}}\left[c_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{-i(p+q)\cdot x} + c_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{-i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{i(p+q)\cdot x}\right] \tag{5.1}$$

Since the integration operator is commutative for independent variables the volume integral reduces the complex integral to Dirac delta functions

$$\int e^{i(p-q)\cdot x}d^3x = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$$

So if we perform q integral any the integral is nonzero only when the q value is equal to p as

$$\int \frac{d^3q}{(2\pi)^3}\frac{q_j}{\sqrt{2E_q}}c_{\boldsymbol{p}}c_{\boldsymbol{q}}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})(2\pi)^3 = \frac{p_j}{\sqrt{2E_p}}c_{\boldsymbol{p}}c_{\boldsymbol{p}}$$

So (5.1) reduces to

$$\begin{aligned}
\int T^{ij}d^3x &= \int \frac{d^3p}{(2\pi)^3}\frac{p^ip_j}{\sqrt{4E_p^2}}\left[c_{\boldsymbol{p}}c_{-\boldsymbol{p}} + c_{\boldsymbol{p}}c^*_{\boldsymbol{p}} + c^*_{\boldsymbol{p}}c_{\boldsymbol{p}} + c^*_{\boldsymbol{p}}c^*_{\boldsymbol{p}}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{p^ip_j}{2E_p}\left[c_{\boldsymbol{p}}c_{-\boldsymbol{p}} + c_{\boldsymbol{p}}c^*_{\boldsymbol{p}} + c^*_{\boldsymbol{p}}c_{\boldsymbol{p}} + c^*_{\boldsymbol{p}}c^*_{\boldsymbol{p}}\right]
\end{aligned}$$

This is the required anisotropic stress required. □

5.3.3. **(SMIN 3.8)** We might suppose, that a vector field has Lorentz-invariant Lagrangian

$$\mathcal{L} = \partial_\mu A^\nu\partial_\nu A^\mu - m^2A_\mu A^\mu$$

(a) Compute the Euler-Lagrange equations of for this Lagrangian.
Solution:
The Euler-Lagrange equation are

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) &= \frac{\partial\mathcal{L}}{\partial A^\nu} \\
\partial_\mu\partial_\nu A^\mu &= -m^2A_\mu \\
\partial^\mu\partial_\nu A^\mu &= -m^2A^\mu
\end{aligned}$$

These are the required Euler-Lagrange equations. □

(b) Assume a plane-wave solution for the vector field

$$A^\mu = \int \frac{d^3p}{(2\pi)^3}\epsilon^\mu \frac{1}{\sqrt{2E_p}}\left[a_p e^{-ip\cdot x} + a_p^* e^{ip\cdot x}\right]$$

where we haven't specified polarization state(s) e^μ explicitly.
Develop an explicit relationship between polarization, the momentum of the field, and the mass. What condition does this impose for a massless vector particle?
Solution:
The stress-energy tensor is

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\alpha)}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L} \\ &= \partial_\alpha A^\mu \partial^\nu A^\alpha - g^{\mu\nu}\left(\partial_\mu A^\alpha \partial_\nu A^\alpha - m^2 A_\alpha A^\alpha\right) \end{aligned}$$

Using the given A^μ vector with every in this expression we get

$$\int \frac{d^3p}{(2\pi)^3}\epsilon^\mu\epsilon^\alpha \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{4E_pE_q}}\left[a_p e^{-ip\cdot x} + a_p^* e^{ip\cdot x}\right]\left[a_q e^{-iq\cdot x} + a_q^* e^{iq\cdot x}\right]$$

Using smiilar development in (5.1) we get

$$\begin{aligned} \int T^{\mu\nu} d^3x &= \int \frac{d^3p}{(2\pi)^3}\epsilon^\mu\epsilon^\nu \frac{1}{\sqrt{4E_pE_p}}\left[(p^\alpha p_\alpha + p^\alpha p_\alpha - m^2)a_p a_p^*\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\epsilon^\mu\epsilon^\nu \frac{2E_p^2 - p^2 - m^2}{2E_p} a_p a_p^* \end{aligned}$$

□

(c) What is the energy density of the vector field?
Solution:
The stress-energy tensor is in the form For energy density $\mu = 0, \nu = 0$

$$T^{00} = \int \frac{d^3p}{(2\pi)^3}\epsilon^0\epsilon^0 \frac{2E_p^2 - 2p^2 - m^2}{2E_p} a_p a_p^*$$

This gives the required energy density. □

5.3.4. **(SMIN 3.9)** We will often describe multiplets of scalar fields,

$$\mathbf{\Phi} = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix},$$

where ϕ_1 and ϕ_2 is each, in this case, a real-valued scalar field for example

$$\mathcal{L}\frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - \frac{1}{2}m^2 \Phi^T \Phi$$

is a compact way of describing two free scalar fields with identical masses. This Lagrangian is symmetric under the transformation

$$\Phi \to (I - i\theta X)\,\Phi$$

where X is some unknown 2×2 matrix, and θ is assumed to be small.

(a) What is the transformation Φ^T? Show that $\Phi^T\Phi$ remains invariant under this transformation.
Solution:
Taking the transpose of Φ we get

$$\Phi^T \to \Phi^T \left(I - i\theta X^T\right)$$

The quantity $\Phi^T\Phi$ after transformation is

$$\begin{aligned}\Phi^T\Phi &\to \Phi^T \left(I - i\theta X^T\right) \cdot (I - i\theta X)\,\Phi \\ &= \left(\Phi^T - i\theta\Phi^T X^T\right)(\Phi - i\theta X\Phi) \\ &= \Phi^T\Phi - i\theta\Phi^T X\Phi - i\theta\Phi^T X^T\Phi - \theta^2\Phi^T XX\Phi \\ &= \Phi^T\Phi - i\theta\Phi^T\left(X + X^T\right)\Phi - \mathcal{O}(\theta^2)\end{aligned}$$

But this transformation preserves the product $\Phi^T\Phi$ only if $X^T = -X$ so that the middle term vanishes.

$$\Phi^T \to \Phi^T(1 + i\theta X)$$

In either of these case

$$\Phi^T\Phi \to \Phi^T\Phi - \mathcal{O}(\theta^2) \approx \Phi^T\Phi$$

This shows that this transformation preserves $\Phi^T\Phi$. □

(b) What is the conserved current in this system?
Solution:
Writing out the Lagrangian in terms of ϕ_1 and ϕ_2 we get

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\begin{pmatrix}\partial_\mu\phi_1 & \partial_\mu\phi_2\end{pmatrix}\begin{pmatrix}\partial^\mu\phi_1 \\ \partial_\mu\phi_2\end{pmatrix} - \frac{1}{2}m^2\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \\ &= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right)\end{aligned}$$

For this transformation the transformed scalar field elements of the matrix are

$$\begin{pmatrix}\phi_1' \\ \phi_2'\end{pmatrix} = \begin{pmatrix}1 - i\theta X_{00} & -i\theta X_{01} \\ -i\theta X_{10} & 1 - i\theta X_{11}\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \begin{pmatrix}(1 - i\theta X_{00})\,\phi_1 & -i\theta X_{01}\phi_2 \\ -i\theta X_{10}\phi_1 & (1 - i\theta X_{11})\,\phi_2\end{pmatrix}$$

Thus the derivative of ϕ_1' and ϕ_2' with θ become

$$\frac{\mathrm{d}\phi_1'}{\mathrm{d}\theta} = -iX_{00}\phi_1 - iX_{01}\phi_2 \qquad \frac{\mathrm{d}\phi_2'}{\mathrm{d}\theta} = -iX_{10}\phi_1 - iX_{11}\phi_2$$

The conserved current now becomes

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\frac{\mathrm{d}\phi_1'}{\mathrm{d}\theta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\frac{\mathrm{d}\phi_2'}{\mathrm{d}\theta} \\ &= \frac{1}{2}\left(\partial^\mu\phi_1\left(-iX_{00}\phi_1 - iX_{01}\phi_2\right) + \partial^\mu\phi_2\left(-iX_{01}\phi_1 - iX_{11}\phi_2\right)\right)\end{aligned} \tag{5.2}$$

This gives explicit expression for conserved current in terms of matrix elements of unknown matrix X. □

(c) As we will see, for the particular case described int his problem, the elements of X are

$$X = \begin{pmatrix}0 & i \\ -i & 0\end{pmatrix}.$$

Compute the conserved current in terms of ϕ_1 and ϕ_2 explicitly.
Solution:
If this matrix is taken then $X_{00} = 0, X_{11} = 0, X_{01} = i$ and $X_{10} = -i$. Substuting these in (5.2) we get,

$$J^\mu = \frac{1}{2}\left[(\partial^\mu \phi_1)\phi_2 - (\partial^\mu \phi_2)\phi_1\right]$$

This gives the explicit expression of conserved current for this particular transformation matrix. □

5.4 Homework Four

5.4.1. **(SMIN 4.1)** Consider a rectangle.

(a) List all the possible unique transformations that can be performed that will leave it looking the same as it did initially.
Solution:
The possible transformations that leave the rectangle looking the same are
i. Leaving where it is (I).
ii. Rotation through 180° (R).
iii. Flipping along the vertical axis through mid points of A & B and C & D (F_y).
iv. Flipping along the horizontal axis through mid points of A & C and B & D (F_x).
□

(b) Construct the multiplication table for your set of transformations.
Solution:
The multiplication table for the transformations is

$\circ$	I	R	F_x	F_y
I	I	R	F_x	F_y
R	R	I	F_y	F_x
F_x	F_x	F_y	I	R
F_y	F_y	F_x	R	I

□

(c) Does this set have the properties of a group?
Solution:
From the multiplication table it is clear that the element satisfy closure. The element I acts as the identity. Each elements are the inverses of themselves. And associativity is evidently followed. This proves that the elements form a group. □

5.4.2. **(SMIN 4.2)** Quaternions are a set of objects that are an extension of imaginary numbers except that there are three of them i, j and k, with the relations

$$i^2 = j^2 = k^2 = ijk = -1$$

.

(a) Construct the smallest group possible that contains all the quarternions.
Solution:
Closure of the group requires that at least, i, j, k and -1 to be the members of the group. Since $i^2 = i \circ i = -1$, i can't be the identity of the group. Similarly j and k can't be identity of the group. That leaves -1 as the only candidate for the identity of the group. If we can satisfy other requirement of group, then i, j, k and -1 will form a group with -1 as the identity.

If we define $-1 \circ -1 = -1$, which doesn't violate any of the given requirements, -1, works as the identity element.

Since $i^2 = i \circ i = -1$ and -1 is identity, i by definition becomes the inverse of itself. Similarly j and k are inverses of themselves. So the group is

$$G\left(\{-1, i, j, k\}, \circ\right)$$

□

(b) Compute the commutation relation $[j, i]$.
Solution:
The commutator of a group is defined as

$$[j, i] = j^{-1} i^{-1} j i$$

Where i^{-1} and j^{-1} are the inverses of i and j respectively. Also since $ijk = -1$. Multiplying by i^{-1} on the left gives $jk = i$ and multiplying by k^{-1} on the right gives $ij = k$. From (5.4.2a) we have $i^{-1} = i$ and $j^{-1} = j$

$$[j, i] = j^{-1} i^{-1} j i = jiji = j(ij)i = j(k)i = (jk)i = ii = -1$$

Since the commutator is identity element of the group, this group is abelian so that the elements commute. □

(c) Construct a multiplication table for the quarternions.
Solution:
The multiplication table becomes

$\circ$	-1	i	j	k
-1	-1	i	j	k
i	i	-1	k	j
j	j	k	-1	i
k	k	j	i	-1

This is the required multiplication table. □

5.4.3. **(SMIN 4.6)** Expand the series $e^{-i\theta\sigma_2}$ explicitly and reduce to common trigonometric, algebraic or hypergeometric functions.
Solution:
The SU(2) rotation matrix with generator σ_2 is $\boldsymbol{M}(\theta) = e^{-i\theta\sigma_2}$. Expanding it out as a Taylor series gives

$$e^{-i\theta\sigma_2} = 1 - i\theta\sigma_2 - \sigma_2^2\frac{\theta^2}{2!} + i\sigma_2^3\frac{\theta^3}{3!} + \sigma_2^4\frac{\theta^4}{4!} - \ldots$$

Since for the Pauli matrices $\sigma_i^2 = 1$ which implies that for odd powers the Pauli matrices are the matrices themselves and for even power they reduce to identity, thus we can write

$$\begin{aligned}
e^{-i\theta\sigma_2} &= 1 - i\theta\sigma_2 - \frac{\theta^2}{2!} + i\sigma_2\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \ldots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots - i\theta\sigma_2 + i\sigma_2\frac{\theta^3}{3!} - \ldots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right) - i\sigma_2\left(\theta - \frac{\theta^3}{3!} + \ldots\right) \\
&= \cos\theta - i\sigma_2\sin\theta
\end{aligned}$$

Writing out the explicit matrix form for identity $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ we get

$$\begin{aligned}\boldsymbol{M} = e^{-i\theta\sigma_2} &= \cos\theta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sin\theta \\ &= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\end{aligned}$$

This is the required 2×2 matrix representation of $e^{-i\theta\sigma_2}$. □

5.4.4. **(SMIN 4.10)** Consider a universe consisting of a complex field defined by two components

$$\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

The Lagrangian takes the form

$$\mathcal{L} = \partial^\mu\Phi^\dagger\partial_\mu\Phi - m^2\Phi^\dagger\Phi.$$

In some sense, there are four fields at work here, ϕ_2, ϕ_2^*, ϕ_1 and ϕ_1^*. But for the purpose of this problem, you should generally think Φ and $\Phi^\dagger$ as representing the two different fields. Since each is a $2-D$ vector, there are still four degrees of freedom.

(a) Consider a rotation in SU(2) in θ^1 direction (σ_x). Expand $\boldsymbol{M}$ as infinite series, and express as a 2×2 matrix of only trigonometric functions of θ^1.
Solution:
The SU(2) rotation matrix with generator σ_x is $\boldsymbol{M}(\theta) = e^{-i\theta\sigma_x}$. Expanding it out as a Taylor series gives

$$e^{-i\theta\sigma_x} = 1 - i\theta\sigma_x - \sigma_x^2\frac{\theta^2}{2!} + i\sigma_x^3\frac{\theta^3}{3!} + \sigma_x^4\frac{\theta^4}{4!} - \dots$$

Since for the Pauli matrices $\sigma_i^2 = 1$ which implies that for odd powers the Pauli matrices are the matrices themselves and for even power they reduce to identity, thus we can write

$$\begin{aligned}e^{-i\theta\sigma_x} &= 1 - i\theta\sigma_x - \frac{\theta^2}{2!} + i\sigma_x\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots - i\theta\sigma_x + i\sigma_x\frac{\theta^3}{3!} - \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) - i\sigma_x\left(\theta - \frac{\theta^3}{3!} + \dots\right) \\ &= \cos\theta - i\sigma_x\sin\theta\end{aligned}$$

Writing out the explicit matrix form for identity $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ we get

$$\begin{aligned}\boldsymbol{M} = e^{-i\theta\sigma_x} &= \cos\theta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin\theta \\ &= \begin{pmatrix}\cos\theta & -i\sin\theta\\ -i\sin\theta & \cos\theta\end{pmatrix}\end{aligned}$$

This is the required 2×2 matrix representation of $SU(2)$ representing rotation in θ^1 direction. □

(b) Verify numerically that your matrix (*i*) is unitary and (*ii*) has a determinant of 1.
Solution:
Checking for Unitarity

$$\begin{aligned}
\boldsymbol{M}\boldsymbol{M}^\dagger &= \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta + (-i\sin\theta)(i\sin\theta) & (i\cos\theta\sin\theta) + (-i\cos\theta\sin\theta) \\ (-i\cos\theta\sin\theta) + (i\cos\theta\sin\theta) & \cos^2\theta + (-i\sin\theta)(i\sin\theta) \end{pmatrix} \\
&= \begin{pmatrix} \sin^2\theta + \cos^2\theta & 0 \\ 0 & sin^2\theta + \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}
\end{aligned}$$

This shows the matrix is unitary. Checking for determinant

$$\det\{\boldsymbol{M}\} = \begin{vmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{vmatrix} = \cos\theta\cos\theta - (-i\sin\theta)(-i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$$

The determinant of the matrix is also 1. □

(c) Compute a general expression for the current associated with the rotations in θ^1.
Solution:
This Lagrangian is clearly invariant under the transformation $\Phi \to \boldsymbol{M}\Phi$. The generator of which is σ_2 thus the conserved current is

$$\mathcal{L} = g^{\mu\nu}\partial_\nu\Phi^\dagger\partial_\mu\Phi - m^2\Phi^\dagger\Phi \qquad \mathcal{L} = \partial^\mu\Phi^\dagger\partial_\mu\Phi - m^2\Phi^\dagger\Phi$$

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi^\dagger)} = g^{\mu\nu}\partial_\mu\Phi = \partial^\nu\Phi \qquad \Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = \partial^\mu\Phi^\dagger$$

$$\begin{aligned}
J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\mathrm{d}\Phi}{\mathrm{d}\epsilon} + \frac{\mathrm{d}\Phi^\dagger}{\mathrm{d}\epsilon}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^\dagger)} \\
&= \partial^\mu\Phi^\dagger(-i\sigma_2\Phi) + (i\sigma_2\Phi^\dagger)\partial^\mu\Phi \\
&= i\sigma_2\left(-\left(\partial^\mu\Phi^\dagger\right)\Phi + \Phi^\dagger\partial^\mu\Phi\right)
\end{aligned}$$

This gives the expression for conserved current. □

5.5 Homework Five

5.5.1. **(SMIN 5.1)** Evaluate

(a) $\{\gamma^0, \gamma^0\}$
Solution:

$$\{\gamma^0, \gamma^0\} = \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2\gamma^0\gamma^0 = 2\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = 2\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 2I_{4\times4}$$

The final matrix is the 4×4 identity matrix □

(b) $\gamma^2\gamma^0\gamma^2$
Solution:

$$\gamma^2\gamma^0\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \gamma^0$$

□

(c) $\left[\gamma^1, \gamma^2\right]$
Solution:

$$\begin{aligned}\left[\gamma^1, \gamma^2\right] &= \gamma^1\gamma^2 - \gamma^2\gamma^1 \\ &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} - \begin{pmatrix} -\sigma_2\sigma_1 & 0 \\ 0 & -\sigma_2\sigma_1 \end{pmatrix} \\ &= \begin{pmatrix} [\sigma_2, \sigma_1] & 0 \\ 0 & [\sigma_2, \sigma_1] \end{pmatrix} = \begin{pmatrix} -2i\sigma_3 & 0 \\ 0 & -2i\sigma_3 \end{pmatrix}\end{aligned}$$

□

5.5.2. **(SMIN 5.3a)** Compute the various traces of the combinations og γ-matrices explicitly

(a) $\mathrm{Tr}\left(\gamma^0\gamma^0\right)$
Solution:

$$\gamma^0\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I_{4\times 4} \Rightarrow \mathrm{Tr}\left(\gamma^0\gamma^0\right) = 4$$

□

(b) $\mathrm{Tr}\left(\gamma^1\gamma^1\right)$
Solution:

$$\gamma^1\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} \Rightarrow \mathrm{Tr}\left(\gamma^1\gamma^1\right) = -4$$

□

(c) $\mathrm{Tr}\left(\gamma^1\gamma^0\right)$
Solution:

$$\gamma^1\gamma^0 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \Rightarrow \mathrm{Tr}\left(\gamma^0\gamma^1\right) = 0$$

□

5.5.3. **(SMIN 5.7)** In quantum field theory calculations, we will often find it useful to cmpute the products like

$$[\bar{u}(1)\gamma^\mu u(2)]\,,$$

where 1 corresponds to spin, mass and 4−momentum of a particle state, and 2 corresponds to similar quantities for second particle. For particle 1. $m = m_1$; $\boldsymbol{p} = 0$, and $s = +1/2$ and for particle 2, $m = 0$; $\boldsymbol{p} = p_z\hat{\mathbf{k}}$ and $s = +1/2$

(a) Calculate the vector values of $[\bar{u}(1)\gamma^\mu u(2)]$ for the states listed.
Solution:
For particle 1 $m = m_1$, $p = 0 \Rightarrow E = \sqrt{p^2 + m_1^2} = m_1$ and for particle 2 $m = 0$, $|\boldsymbol{p}| = p_z \Rightarrow E = \sqrt{p^2 + m_1^2} = p$, And $[\bar{u}(1)\gamma^\mu u(2)] = u^\dagger(1)\gamma^0\gamma^\mu u(2)$ so we we have

$$u(1) = \frac{m_1}{\sqrt{E+0}} \begin{pmatrix} 1 \\ 0 \\ \frac{E}{m_1} \\ 0 \end{pmatrix} \qquad \Rightarrow u^\dagger(1) = \sqrt{m_1} \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$$

$$u(2) = \frac{m}{\sqrt{E+p}} \begin{pmatrix} 1 \\ 0 \\ \frac{E+p}{m} \\ 0 \end{pmatrix} = \begin{pmatrix} m/\sqrt{E+p} \\ 0 \\ \sqrt{E+p} \\ 0 \end{pmatrix} \qquad \lim_{m\to 0} u(2) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2E} \\ 0 \end{pmatrix}$$

Alos the various product of gamma matrices are

$$\gamma^0\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \gamma^0\gamma^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

$$\gamma^0\gamma^2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \qquad \gamma^0\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

Using these to calculate the vectors we get
the various components are

$$u^\dagger(1)\gamma^0\gamma^0 u(2) = \sqrt{2m_1E} \qquad u^\dagger(1)\gamma^0\gamma^2 u(2) = 0$$
$$u^\dagger(1)\gamma^0\gamma^3 u(2) = \sqrt{2m_1E} \qquad u^\dagger(1)\gamma^0\gamma^1 u(2) = 0$$

So the required matrix is

$$[\bar{u}(1)\gamma^\mu u(2)] = \begin{pmatrix} \sqrt{2m_1E} \\ 0 \\ 0 \\ \sqrt{2m_1E} \end{pmatrix}$$

□

(b) Do the same for spin down states.
Solution:
Similarly for the spin down states we get

$$u(1) = \frac{m_1}{\sqrt{E-0}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{E}{m_1} \end{pmatrix} \qquad \Rightarrow u^\dagger(1) = \sqrt{m_1} \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$$

$$u(2) = \frac{m}{\sqrt{E-p}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{E-p}{m} \end{pmatrix} = \begin{pmatrix} 0 \\ m/\sqrt{E-p} \\ 0 \\ \sqrt{E-p} \end{pmatrix} \qquad \lim_{m\to 0} u(2) = \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix}$$

Similarly we get

$$[\bar{u}(1)\gamma^\mu u(2)] = \begin{pmatrix} \sqrt{2m_1E} \\ 0 \\ 0 \\ \sqrt{2m_1E} \end{pmatrix}$$

□

(c) Calculate the vector of values for $s_2 = -1/2$

5.5.4. **(SMIN 5.12)** For the single-particle Dirac equation Hamiltonian

$$\hat{H} = -i\gamma^i \partial_i + m$$

(a) Compute the commutator of Hamiltonian operator with the z component of the angular momentum operator $\left[\hat{H}, \hat{L}_2\right]$, where

$$\hat{\boldsymbol{L}} \equiv \boldsymbol{r} \times \boldsymbol{p}$$

Solution:
Writing $-i\partial_i = p_i$ we get

$$\hat{H} = -i\gamma^i \partial_i + m = \gamma^i p_i$$

Since m is scalar it commutes with the L operator so we get

$$\left[\hat{H}, \hat{L}_z\right] = \left[\gamma^i p_i, L_z\right] = \left[\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z, xP_x - yP_x\right]$$

But using the commutation relations $[x_i, p_j] = i\delta_{ij}$ and $[p_i, p_j] = 0$ we get

$$\begin{aligned}
\left[\gamma^1 p_x, xP_y - yP_x\right] &= \left[\gamma^1 p_x, xp_y\right] - \left[\gamma^1 p_x, yp_x\right] = \gamma^1(-ip_y) = -i\gamma^1 p_y \\
\left[\gamma^2 p_y, xP_y - yP_x\right] &= \left[\gamma^2 p_y, xp_y\right] - \left[\gamma^2 p_y, yp_x\right] = \gamma^2(ip_x) = i\gamma^2 p_x \\
\left[\gamma^3 p_z, xP_y - yP_x\right] &= 0
\end{aligned}$$

Thus the commutation becomes

$$\left[\hat{H}, \hat{L}_z\right] = -i\gamma^1 p_y + i\gamma^2 P_x$$

Which is the required commutation relation of Hamiltonian and the z component of L. □

(b) now consider the spin operator

$$\hat{\boldsymbol{S}} = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Compute the z component of $\hat{S}^z u_-(p)$
Solution:
The operator $\hat{S}^z$ and the state $u_-(p)$ are

$$\hat{S}^z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad u_-(p) = \frac{m}{\sqrt{E-p}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{E-p}{m} \end{pmatrix}$$

Apply ying the operator simply scales by $\frac{1}{2}$ and flips the sign of secon and last component yielding

$$\hat{S}^z u_-(p) = -\frac{1}{2} u_-(p)$$

This is the required state after operation. □

(c) Compute $\left[\hat{H}, \hat{S}_2\right]$.
Solution:
Writing Hamiltonian as

$$\hat{H} = -i\gamma^i\partial_i + m = \gamma^i p_i$$

Since the operator p_i commute with the 4×4 matrices S and γ

$$\left[\gamma^i p_i, S_z\right] = \left[\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z, S_z\right] = p_x\left[\gamma^1, S_z\right] + p_y\left[\gamma^2, S_z\right] + p_z\left[\gamma^3, S_z\right]$$

Using the commutation relations $\left[S_z, \gamma^1\right] = i\gamma^2$ and $\left[S_z, \gamma^2\right] = -i\gamma^1$ we obtain

$$\left[\hat{H}, \hat{S}_z\right] = -i\gamma^2 p_x + i\gamma^1 p_y$$

Which is the required commutation relation. □

(d) Coomparint your answers, derive a conserved quantity for the free fermions.
Solution:
Clearly from two parts above $\left[\hat{H}, \mathbf{L}+\mathbf{S}\right] = 0$ thus the conserved operator is $\mathbf{L}+\mathbf{S}$. For free fermion of state $\psi(p)$ the conserved quantity is

$$(\mathbf{L}+\mathbf{S})\psi(p)$$

The eigenvalue of this operator gives the conserved quantity. □

5.6 Homework Six

5.6.1. **(SMIN 6.2)** Suppose, contrary to our work in this chapter, that the photon had a very small mass, 10^{-4} eV. What would the effective range of the electromagnetic force be? Express your answer in meters. Approximately how light (in kilograms) would the photon need to be such that earth-scale magnetic fields would still be measurable?
Solution:
The interaction field is approximately given by

$$E_{\text{int}} \approx \frac{e^{-mr}}{4\pi r}$$

For a measuralble field $E \approxeq 1$ so with $M = 10^{-4}\text{eV} \approx 1.8 \times 10^{-40} kg$ we have

$$\frac{e^{-mr}}{4\pi r} \approxeq 1 \qquad \Rightarrow \qquad r \approx 0.0795m$$

The magnetic field of earth is $B = 25 \times 10^{-9}T$ for this to be measurable in earth scale $r \approx 6.4 \times 10^6 m$ we again solve for m in the equation

$$B_{\text{int}} \approx \frac{e^{-mr}}{4m\pi r}$$
$$25 \times 10^{-9} \approx \frac{e^{-m6.4\times10^6}}{4m\pi 6.4 \times 10^6} \qquad \Rightarrow m \approx 1.9 \times 10^{-7} kg$$

The mass of photon has to be very low in order for this to be measured. □

5.6.2. **(SMIN 6.6)** In classical electrodynamics, radiation is propagated along the Poynting vector,

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B},$$

an ordinary 3-vector. Express the components of S^i in terms of components of $F^{\mu\nu}$ in as simplified form as possible.
Solution:
In index notation the cross products of two vector is

$$S^i = \varepsilon_{ijk} E^j B^k$$

Since the magnetic field and electric field components in terms of the Farady tensor elements are

$$F^{0i} = E^i \qquad F^{ij} = B^k$$

The Poynting vector becomes

$$S^i = \varepsilon_{ijk} F^{0j} F^{ij} \qquad \Rightarrow \qquad \boldsymbol{S} = \begin{pmatrix} F^{02}F^{12} - F^{03}F^{31} \\ F^{03}F^{23} - F^{01}F^{12} \\ F^{01}F^{31} - F^{02}F^{23} \end{pmatrix}$$

This is the required Poynting vector in terms of the components of Farady tensor. □

5.6.3. **(SMIN 6.9)** In developing the two polarization-states model for the photon we lied upon $U(1)$ gauge invariance, which in turn depends on a massless photon. We know that a spin-1 particle are supposed to have three spin states, but we claimed that the third state was swallowed by the Coulomb gauge condition. Lets' approach the question of three states by assuming that the photon does have mass and obeys lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A^\mu A_\mu$$

(a) Write the Euler-Lagrange equation for the massive photon field.
Solution:
Since by definition the Farady tensor is the antisymmetric tensor formed by various derivatives of the components of A^μ.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad\qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

The product term in the lagrangian is:

$$\begin{aligned} F^{\mu\nu} F_{\mu\nu} &= (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \partial^\mu A^\nu \partial_\mu \partial_\nu - \partial^\mu A^\nu \partial_\nu A_\mu - \partial^\nu A^\mu \partial_\mu A_\nu + \partial^\nu A^\mu \partial_\nu A_\mu \\ &= 2(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) \end{aligned}$$

Writing out the lagrangian in terms of these components we get

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) + \frac{1}{2} M^2 A^\mu A_\mu$$

Thus the Euler-Lagrange equations become

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) &= \frac{\partial \mathcal{L}}{\partial A_\nu} \\ -\frac{1}{2} \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) &= \frac{1}{2} M^2 A^\nu \\ -\partial_\mu F^{\mu\nu} &= M^2 A^\nu \end{aligned}$$

These are the required Euler-Lagrange equations. □

(b) Let the photon field take the form of a single plane wave:

$$A^\mu = \varepsilon^\mu e^{-ip\cdot x}.$$

Express the Euler-Lagrange equations as dot products of p and ε with themselves and with each other. Show that the transverse wave condition drops out of the dispersion relation regardless of whether the field has mass.
Solution:
For this field the Farady tensor becomes

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -ip_\mu \varepsilon^\nu e^{-ip\cdot x} + ip_\nu \varepsilon^\mu e^{-ip\cdot x} = -i\left(p_\mu \varepsilon^\nu - p_\nu \varepsilon^\mu\right) e^{-ip\cdot x}$$

So the Euler-Lagrange equations become

$$\begin{aligned} -\partial_\mu F^{\mu\nu} &= M^2 A^\nu \\ -\left[-i\left(p_\mu \varepsilon^\nu - p_\nu \varepsilon^\mu\right)(-ip^\mu) e^{-ip\cdot x}\right] &= M^2 e^\nu e^{-ip\cdot x} \\ \left(p^\mu p_\mu \varepsilon^\nu - p_\nu \varepsilon^\mu p^\mu\right) &= M^2 \varepsilon^\nu \\ \left(p\cdot p\varepsilon^\nu - p_\nu \varepsilon\cdot p\right) &= M^2 \varepsilon^\nu \end{aligned}$$

Regardless of the mass the coefficient of p_ν on the LHS must be 0 so the dot product $\varepsilon\cdot p = 0$. □

(c) What is the third possible polarization-state for a massive photon propagating in the z-direction?
Solution:
For this vector field, $p\cdot p = M^2$ and $\varepsilon\cdot p = 0$. For a particle moving in z direction with moentum p_z and Energy E the momentum 4-vector is $p^\mu = \begin{pmatrix} E & 0 & 0 & p_z \end{pmatrix}^T$. The linearly independent ε vector satisfying these relations apart from the ones given is

$$\varepsilon_3 = \begin{pmatrix} p_z \\ 0 \\ 0 \\ E \end{pmatrix} \qquad \text{as} \qquad \varepsilon_3 \cdot p = \begin{pmatrix} p_z \\ 0 \\ 0 \\ E \end{pmatrix} \cdot \begin{pmatrix} E \\ 0 \\ 0 \\ p_z \end{pmatrix} = p_z E - E p_z = 0$$

Since the inner product of ε with itself is $p^2 - E^2 = -M^2$, we could choose normalization factor i/M for ε. □

(d) What are the electric and magnetic fields of the massive photon field in this third polarization state? What happens to theose fields for $m = 0$?
Solution:
Now the Electric and magnetic fields are simply the components of Farady tensor

$$\begin{aligned} E^i &= F^{0i} = -i\left(p_0\varepsilon^i - p_i\varepsilon^0\right) e^{ip\cdot x} \\ E_x &= F^{01} = -i\left(p_0\varepsilon^1 - p_1\varepsilon^0\right) e^{ip\cdot x} = 0 \\ E_y &= F^{02} = -i\left(p_0\varepsilon^2 - p_2\varepsilon^0\right) e^{ip\cdot x} = 0 \\ E_z &= F^{03} = -i\left(p_0\varepsilon^3 - p_3\varepsilon^0\right) e^{ip\cdot x} = -i(E^2 - p^2)e^{-ip\cdot x} = -iM^2 e^{-ip\cdot x} \end{aligned}$$

$$\begin{aligned} B_k &= F^{ij} = -i\left(p_i\varepsilon^j - p_i\varepsilon^j\right) e^{ip\cdot x} \\ B_x &= F^{23} = -i\left(p_2\varepsilon^3 - p_3\varepsilon^2\right) e^{ip\cdot x} = 0 \\ B_y &= F^{31} = -i\left(p_3\varepsilon^1 - p_1\varepsilon^4\right) e^{ip\cdot x} = 0 \\ B_z &= F^{12} = -i\left(p_1\varepsilon^2 - p_2\varepsilon^1\right) e^{ip\cdot x} = 0 \end{aligned}$$

So $\boldsymbol{E} = \left[-iM^2 e^{-ip\cdot x}\right]\hat{\mathbf{z}}$ and $\boldsymbol{B} = 0$. If $M = 0$ then the Electric field vanishes as well, so both the fields vanish. □

5.6.4. **(SMIN 6.10)** Consider an electron in a spin state

$$\phi = \begin{pmatrix} a \\ b \end{pmatrix}$$

in a magnetic field B_0 oriented along the z-axis. We will calculate the *Larmor Frequency* by which the electron precesses.

(a) Turn the interaction Hamiltonian into a first order differential equation in time.
Solution:
The given interaction hamiltonian is

$$\hat{H}_{\text{int}} = -\frac{q_e B_0}{2m} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Writing the hamiltonian as $i\partial_0 = i\frac{\partial}{\partial t}$

$$i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{q_e B_0}{2m} \begin{pmatrix} a \\ -b \end{pmatrix}$$

So the differential equations are

$$i\frac{\partial a}{\partial t} = -\frac{q_e B_0}{2m} a \qquad i\frac{\partial b}{\partial t} = \frac{q_e B_0}{2m} b$$

These are the required differential equations. □

(b) Solve the differential equation in part a. What is the Frequency of oscillation of the phase *difference* between teh two components?
Solution:
The solutions are

$$a = a_0 e^{\frac{iq_e B_0}{2m} t} \qquad b = b_0 e^{-\frac{iq_e B_0}{2m} t}$$

The phase difference is

$$\varphi = \left(\frac{iq_e B_0 t}{2m}\right) - \left(-\frac{iq_e B_0 t}{2m}\right) = \frac{iq_e B_0 t}{m}$$

The frequency of oscillation is

$$\frac{iq_e B_0}{m}$$

This is the required frequency. □

Chapter 6

Statistical Mechanics

6.1 Homework One

6.1.1. A particular system obeys two equations of state

$$T = \frac{3As^2}{v} \qquad \text{(thermal equation of state)}, \qquad P = \frac{As^3}{v^2} \qquad \text{(mechanical equation of state)}.$$

Where A is a constant.

(a) Find μ as a function of s and v, and then find the fundamental equation.
Solution:
Given P and T the differential of each of them can be calculated as

$$\begin{aligned} dT &= \frac{6As}{v}ds - \frac{3As^2}{v^2}dv \qquad \Rightarrow \qquad sdT = \frac{6As^2}{v}ds - \frac{3As^3}{v^2}dv \\ dP &= \frac{3As^2}{v^2}ds - \frac{2As^3}{v^3}dv \qquad \Rightarrow \qquad vdP = \frac{3As^2}{v}ds - \frac{2As^3}{v^2}dv \end{aligned}$$

The Gibbs-Duhem relation in energy representation allows to calculate the value of μ.

$$\begin{aligned} d\mu &= vdP - sdT \\ &= \frac{3As^2}{v}ds - \frac{2As^3}{v^2}dv - \frac{6As^2}{v}ds + \frac{3As^3}{v^2}dv \\ &= -\left[\frac{3As^2}{v}ds - \frac{As^3}{v^2}dv\right] = -d\left(\frac{As^3}{v}\right) \end{aligned}$$

This can be identified as the total derivative of $\frac{As^3}{v}$ so the rel

$$d\mu = -d\left(\frac{As^3}{v}\right) \qquad \Rightarrow \qquad \mu = -\frac{As^3}{v} + k$$

Where k is arbitrary constant. We can plug this back to Euler relation to find the fundamental equation as.

$$\begin{aligned} u &= Ts - Pv + \mu \\ &= \frac{3As^3}{v} - \frac{As^3}{v} - \frac{As^3}{v} + k = \frac{As^3}{v} + k \end{aligned}$$

So the fundamental equation of the system is $\frac{As^3}{v} + k$. □

(b) Find the fundamental equation of this system by direct integration of the molar form of the equation.
Solution:
The differential form of internal energy is

$$\begin{aligned} du &= Tds - Pdv \\ &= \frac{3As^2}{v}ds - \frac{As^3}{v^2}dv \end{aligned}$$

As before this is just the total differential of $\frac{As^3}{v}$ so the relation leads to

$$du = d\left(\frac{As^3}{v^2}\right) \qquad \Rightarrow \qquad u = \frac{As^3}{v} + k$$

This k should be the same arbitrary constant that we got in the previous problem. □

6.1.2. The fundamental equation of system A is

$$S = C(NVE)^{1/3},$$

and similarly for system B. The two system are separated by rigid, impermeable, adiabatic wall. System A has a volume of $9 \times 10^{-6} m^3$ and a molenumber of 3 moles. System B has volume of $4 \times 10^{-6} m^3$ and a mole number of 2 moles. The total energy of the composite system is $80J$.

(a) Plot the entropy as a function of $E_A/(E_A + E_B)$.
Solution:
Since the total energy of the system is $80J$ the sum $E_A + E_B = 80J$. The total entropy of system can be written as

$$S = C\left[\left\{N_1V_1 \cdot 80 \cdot \left(\frac{E_A}{E_A + E_B}\right)\right\}^{\frac{1}{3}} + \left\{N_2V_2 \cdot 80 \cdot \left(1 - \frac{E_A}{E_A + E_B}\right)\right\}^{\frac{1}{3}}\right]$$

The graph of Entropy S vs the energy fraction is shown in Figure 6.1. □

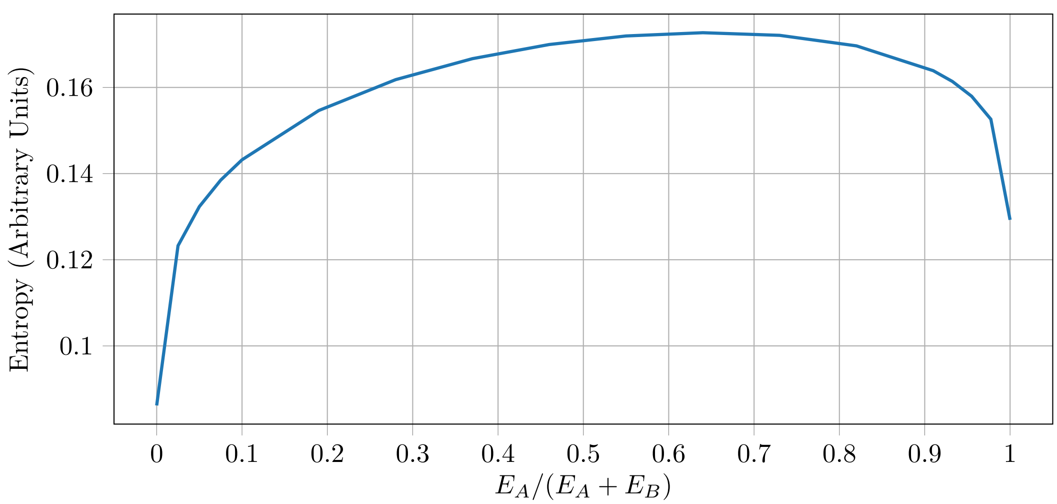


Figure 6.1: Plot of Entropy vs energy fraction.

(b) If the internal wall is now made diathermal and the system is allowed to come to equilibrium, what are the internal energies of each of the individual systems?

Solution:
If the wall is made diathermal and the energy can flow the total energy of the remains constant $E = E_A + E_B$. Taking differential on both sides we get $dE = dE_A + dE_B = 0$. Since there is no change in volume or the number of molecules $dV = 0$ and $dN = 0$. Thus the differential relation of entropy reduces to $dS = \frac{1}{T}dE$. The additive property allows us to write

$$dS = dS_A + dS_B = \frac{1}{T_A}dE_A + \frac{1}{T_B}dE_B \qquad \Rightarrow \frac{dE_A}{T_A} = -\frac{dE_B}{T_B} \quad \Rightarrow \quad \frac{1}{T_A} = \frac{1}{T_B}$$

The quantities T_A and T_B for each systems can be from the fundamental equation thus

$$\frac{1}{T_A} = \left(\frac{\partial S_A}{\partial E_A}\right) = \frac{C}{3}\left(\frac{N_A V_A}{E_A^2}\right)^{1/3} \qquad \frac{1}{T_B} = \left(\frac{\partial S_B}{\partial E_B}\right) = \frac{C}{3}\left(\frac{N_B V_B}{E_B^2}\right)^{1/3}$$

These expressions can be simplified down to get and noting $E_A + E_B = 80J$ we have two linear expressions

$$E_B = \sqrt{\frac{N_B V_B}{N_A V_A}} E_A \qquad E_A + E_B = 80 \qquad \Rightarrow \qquad E_A = 51.93J \qquad E_B = 28.07J$$

So the after equilibrium the internal energy of system A is $E_A = 51.93J$ and for system B it is $E_B = 28.07J$. □

(c) Comment on the relation between these two results.
Solution:
The graph of S vs $E_A/(E_A + E_B)$ is skewed to the right and its maximum is at $E_A/(E_A + E_B) = 0.64$. The final energy of system A is 51.93 which is $0.64 \cdot 80$. Thus the final final energies are such that the total final entropy is maximum. □

6.1.3. An impermeable, diathermal, and rigid partition divides a container into two subvolumes, of volume nV_0 and mV_0. The subvolumes contain respectively, n moles of H_2 and m moles of Ne, each to be considered as a simple ideal gas. The system is maintained at a constant temperature T. The partition is suddenly ruptured and equilibrium is allowed to re-establish. Find the change in entropy of the system. How is the result related to the "entropy of mixing"?
Solution:
The fundamental equation of ideal gas can be written as $U = cNRT$ and equivalently as $PV = NRT$ thus the quantities

$$U = cNRT \qquad \Rightarrow \qquad \frac{1}{T} = cR\frac{N}{U} = \frac{cR}{u}$$
$$PV = NRT \qquad \Rightarrow \qquad \frac{P}{T} = R\frac{N}{V} = \frac{R}{v}$$

Sine it is true for each of these systems we can write

$$ds = \frac{1}{T}du + \frac{P}{T}dv \quad \Rightarrow \quad ds = \frac{cR}{u}du + \frac{R}{v}dv \quad \Rightarrow \quad s = s_0 + cR\ln\left(\frac{u_f}{u_i}\right) + R\ln\left(\frac{v_f}{v_i}\right)$$

The initial and final molar volume for each of the gases is

$$v_{ih} = \frac{nV_0}{n} = V_0 \qquad v_{in} = \frac{mV_0}{m} = V_0$$
$$v_{fh} = \frac{(m+n)V_0}{n} = \left(1 + \frac{m}{n}\right)V_0 \qquad v_{fn} = \frac{(m+n)V_0}{m} = \left(1 + \frac{n}{m}\right)V_0$$

Also since the temperature of system is constant and that no heat flows in or out of the composite system the change in internal energy is zero thus $u_i = u_f$ for both thus the total final entropy become

$$s_h = s_{0h} + cR\ln\left(\frac{u_{fh}}{u_{ih}}\right) + R\ln\left(\frac{v_{fh}}{v_{ih}}\right) = s_{0h} + R\ln\left(1+\frac{m}{n}\right)$$

Similarly for Ne the final entropy of system is

$$s_n = s_{0n} + R\ln\left(1+\frac{n}{m}\right)$$

The total change in entropy is

$$\begin{aligned}\Delta S &= ms_n + ns_h - (ms_{0n} + ns_{0h})\\ &= mR\ln\left(1+\frac{n}{m}\right) + ms_{n0} + nR\ln\left(1+\frac{m}{n}\right) + ns_{h0} - (ms_{0n} + ns_{0h})\\ &= n\ln\left(1+\frac{m}{n}\right) + m\ln\left(1+\frac{n}{m}\right)\end{aligned}$$

This is exactly equal to the Entropy of mixing. □

6.1.4. The entropy of blackbody radiation is given by the formula

$$S = \frac{4}{3}\sigma V^{1/4}E^{3/4},$$

where σ is a constant.

(a) Determine the temperature and the pressure of the radiation.
Solution:
The Entropy relation can be inverted to get

$$E = \left(\frac{81S^4}{256\sigma^4 V}\right)^{1/3}$$

Differentiating this with repsect to V to get the pressure gives

$$P = -\left(\frac{\partial E}{\partial V}\right) = \frac{\sqrt[3]{6}S^{\frac{4}{3}}}{8V^{\frac{4}{3}}\sigma^{\frac{4}{3}}}$$

The temperature similarly is

$$T = \left(\frac{\partial E}{\partial S}\right) = \frac{\sqrt[3]{6}\sqrt[3]{S}}{2\sqrt[3]{V}\sigma^{\frac{4}{3}}}$$

Thus the temperature and pressure are determined. □

(b) Prove that

$$PV = \frac{E}{3}$$

Solution:
Substuting $S = \frac{4}{3}\sigma V^{1/4}E^{3/4}$ into the pressure expression

$$P = \frac{\sqrt[3]{6}S^{\frac{4}{3}}}{8V^{\frac{4}{3}}\sigma^{\frac{4}{3}}} = \frac{\sqrt[3]{6}\left(\frac{4}{3}\sigma V^{1/4}E^{3/4},\right)^{\frac{4}{3}}}{8V^{\frac{4}{3}}\sigma^{\frac{4}{3}}} = \frac{E}{3V} \qquad \Rightarrow PV = \frac{E}{3}$$

Thus $PV = \frac{E}{3}$ is proved as required. □

6.1.5. For a particular system, it is found that $e = (3/2)Pv$ and $P = AvT^4$. Find the molar Gibbs potential and molar Helmholtz potential for the system.

Solution:

Since there are two equations of state we can modify them to express the intensive parameters as

$$P = \frac{2e}{3v} \qquad T = \left(\frac{P}{Av}\right)^{1/4} = \left(\frac{2u}{3Av^2}\right)^{1/4}$$

These can be used in Entropy differential equation to get

$$\begin{aligned} ds &= \frac{1}{T}de + \frac{P}{T}dv \\ &= \left(\frac{3A}{2}\frac{v^2}{e}\right)^{1/4} de + \left(\frac{8A}{27}\frac{e^3}{v^2}\right)^{1/4} dv \end{aligned}$$

The above expression can be recocnized as the total differential of $\left(\frac{128Av^2e^3}{27}\right)^{1/4}$

$$ds = d\left(\frac{128Av^2e^3}{27}\right)^{1/4} \qquad \Rightarrow \qquad s = \left(\frac{128Av^2e^3}{27}\right)^{1/4} + s_0$$

Multiplying thourgh by N to get the non molar quantities we get

$$S = \left(\frac{128A}{27}\frac{V^2E^3}{N}\right)^{1/4} + S_0$$

This above relation can be inverted to get the fundamental energy representation. So we get

$$E = \left[\frac{27}{128A}\frac{N}{V^2}(S - S_0)^4\right]^{1/3}$$

This serves as the fundamental Energy relation which can be used to find the Gibbs and Helmholtz potential.

We can now find the intensive parameters T and P in terms of the extensive parameters as

$$T = \left(\frac{\partial E}{\partial S}\right) = \frac{\partial}{\partial S}\left[\frac{27}{128A}\frac{N}{V^2}(S - S_0)^4\right]^{1/3} = \frac{1}{\sqrt[3]{2A}}\left(\frac{N}{V^2}\right)^{1/3}(S - S_0)^{1/3}$$

We can invert to find S as a function of T so

$$S = S_0 + \frac{2AV^2}{N}T^3$$

Similarly we can find the intensive parameter P as

$$P = -\left(\frac{\partial E}{\partial V}\right) = \frac{1}{2\sqrt[3]{2A}}N^{1/3}\left(\frac{S - S_0}{V^5}\right)^{1/3}$$

This can again be inverted to get V as a function of P

$$V = \left[\frac{1}{16A}\frac{N(S - S_0)^4}{P^3}\right]^{1/5}$$

Equipped with these functions we can now find the Gibbs potential as

$$\begin{aligned} G &= E - TS + PV \\ &= \left[\frac{27}{128A}\frac{N}{V^2}(S - S_0)^4\right]^{1/3} - T\cdot\left(S_0 + \frac{2AV^2}{N}T^3\right) + P\cdot\left[\frac{1}{16A}\frac{N(S - S_0)^4}{P^3}\right]^{1/5} \\ &= \left(\frac{A^3P^2}{N^3}T^{12}V^8\right)^{1/5} - \frac{AT^4V^2}{2N} \end{aligned}$$

This gives the Gibbs Potential now the Helmholtz potential can be similarly found as

$$\begin{aligned} F &= E - TS \\ &= \left[\frac{27}{128A}\frac{N}{V^2}(S-S_0)^4\right]^{1/3} - T\cdot\left(S_0 + \frac{2AV^2}{N}T^3\right) \\ &= -\frac{AT^4V^2}{2N} \end{aligned}$$

Thus the Helmholtz potential is $-\frac{AT^4V^2}{2N}$. □

6.2 Homework Two

6.2.1. Show that for a given N_r with $\sum_i^N p_i = 1$, the uncertainty function $S(\{p_i\})$, takes its maximum value when $p_i = \frac{1}{N}$ for all i, that is $S(\{p_i\}) = A(N)$

Solution:

The uncertainty function is $S(\{p_i\}) = -C\sum_i p_i \ln p_i$. We want to maximize this function subject to the constraint $\sum_i p_i = 1$. Using Lagrange's multiplier method to find the extremum of function, we can define a new function $S - \lambda(\sum_i p_i - 1)$

$$\begin{aligned} \frac{\partial S'}{\partial p_j} &= \frac{\partial}{\partial p_j}\left[-C\sum_i p_i \ln p_i - \lambda\left(\sum_i p_i - 1\right)\right] \\ &= -C\sum_i\left(\delta_{ij}\ln p_i + \frac{1}{p_j}p_i\delta_{ij}\right) - \lambda\left(\sum_i \delta_{ij}\right) \\ &= -C(\ln p_j + 1) - \lambda \end{aligned} \tag{6.1}$$

But for extremum condition of this function the partial derivative with respect to every p_j should vanish. Thus we get

$$\ln p_j = -\frac{\lambda}{C} - 1 \qquad \Rightarrow \qquad p_j = \exp\left[-\frac{\lambda}{C} - 1\right]$$

The RHS of above expression is a constant, lets call that constant M so $p_i = M$ for some constant M but since probability has to add 1 we get

$$\sum_j p_j = 1; \qquad \Rightarrow \sum_j M = 1 \qquad \Rightarrow MN = 1 \qquad \Rightarrow M = \frac{1}{N}$$

Substuting this back we get

$$p_j = \frac{1}{M}$$

Thus the uncertainty function takes it maximum value when $p_i = 1/N$ for all p_i □

6.2.2. Consider a urn problem discussed in class: An urn is filled with balls, each numbered $n = 0, 1, 2 \ldots$. The average value of n is $\langle n\rangle = 2/7$. Calculate the probabilities p_0, p_1 and p_2 which yield the maximum uncertainty. Find the expectation value, based on these probabilities $\langle n^3\rangle - 2\langle n\rangle$.

Solution:

The expectation value of n is given by

$$\langle n\rangle = p_0\cdot 0 + p_1\cdot 1 + p_2\cdot 2 \quad \Rightarrow \quad p_1 + 2p_2 = 2/7$$

This is one of the constraints for maximizing the uncertainty function, the other constraint equation is $p_0 + p_1 + p_2 = 1$. Using these as we calculated in (6.2) we have

$$S' = \frac{S}{C} - \alpha(p_1 + 2p_2 - 2/7) - \beta(p_0 + p_1 + p_2 - 1)$$

Taking derivative with respect to α and β and equating to zero gives

$$\begin{aligned}\ln p_0 + 1 - \beta = 0\\ \ln p_1 + 1 - \alpha - \beta = 0\\ \ln p_2 + 1 - 2\alpha - \beta = 0\end{aligned}$$

These three equations along with two constraint equation form five equation in five unkonwn $p_0, p_1, p_2, \alpha, \beta$. We can solve this equation to get the numeric value of the parameters. Solving for the parameters we get

$$p_0 = \frac{15}{21} \qquad p_1 = \frac{4}{21} \qquad p_2 = \frac{1}{21}$$

Now the requried function is

$$\begin{aligned}\langle n^3 \rangle - 2\langle n \rangle &= p_0 \cdot 0 + p_1 \cdot 1^3 + p_2 \cdot 2^3 - \langle n \rangle\\ &= p_1 + 8p_2 - 2\frac{2}{7}\\ &= \frac{4}{21} + 8\frac{1}{21} - \frac{4}{7}\\ &= 0\end{aligned}$$

The requried value is 0

□

6.2.3. Assuming the entropy, S and the number of microstates, Ω of a physical system are related through an arbitrary functional form $S = f(\Omega)$, show that the additive character of S (extensive parameter) and the multiplicative parameter Ω meaning $\Omega = \Omega_1, \Omega_2, \ldots$, is the number of microscopic states for a subsystem necessarily require that the function $F(\omega)$ is of the form

$$S = k\ln(\Omega)$$

where k is a (universal) constant. The form was first written down by Max Plank.
Solution:
Given the multiplicative parameter $\Omega = \Omega_1 \cdot \Omega_2 \ldots \Omega_r$. The extensive parameter as a function of this parameter which is a additive function be S. Thus we have

$$\begin{aligned}S(\Omega_1 \cdot \Omega_2 \ldots \Omega_r) = S(\Omega_1) + S(\Omega_1) + \ldots + S(\Omega_r)\\ S(\Omega) = \sum_j^r S(\Omega_j)\end{aligned}$$

Differentiating with respect to Ω_i on both sides

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\Omega_i} S(\Omega) = \frac{\mathrm{d}}{\mathrm{d}\Omega_i}\sum_j^r S(\Omega_j)\\ \frac{\mathrm{d}S(\Omega)}{\mathrm{d}\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega_i} = \sum_j^r \frac{\mathrm{d}S(\Omega_j)}{\mathrm{d}\Omega_i}\delta_{ij}\end{aligned}$$

But since the derivative of product $\Omega = \Pi_j \Omega_j$ with respect to Ω_i is just the product without that parameter $\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega_i} = \Pi_{j\neq i}\Omega_j$. Multiplying both sides by Ω_i we get

$$\Omega_i \left(\Pi_{j\neq i}\Omega_j\right) \frac{\mathrm{d}S(\Omega)}{\mathrm{d}\Omega} = \Omega_i \frac{\mathrm{d}S(\Omega_i)}{\mathrm{d}\Omega_i} \qquad \Rightarrow \qquad \Omega\frac{\mathrm{d}S(\Omega)}{\mathrm{d}\Omega} = \Omega_i \frac{\mathrm{d}S(\Omega_i)}{\mathrm{d}\Omega_i}$$

But the expression $\frac{1}{x}dx \equiv d(\ln(x))$ recognizing similar expression in both sides of the equality above we get

$$\frac{\mathrm{d}S(\Omega)}{\mathrm{d}(\ln\Omega)} = \frac{\mathrm{d}S(\Omega_i)}{\mathrm{d}(\ln\Omega_i)}$$

The expression in RHS is independent of expression on right. Since the product of the parameters can be varied while still keeping one of the parameters Ω_i constant. So the expression can only be equal to each other if they are equal to a constant.

$$\frac{\mathrm{d}S(\Omega)}{\mathrm{d}(\ln\Omega)} = k \qquad \Rightarrow \qquad dS(\Omega) = kd(\ln\Omega)$$

Integrating this expression we get

$$S(\Omega) = k\ln\Omega$$

Which is the required expression. □

6.2.4. Show that in $\ln x \leq x - 1$, if for all real positive x. The equality holds for $x = 1$.
Solution:
Rearranging the equation $\ln x - x \leq -1$. Let us define a function $g(x) = \ln x - x$. Differentiating this function with respect to x we get

$$g'(x) = \frac{1}{x} - 1 = \frac{1-x}{x} = -\frac{x-1}{x}$$

Since for all positive values of x i.e., $\forall x > 0$ we have

$$x - 1 < x \Rightarrow \frac{x-1}{x} < 1 \qquad \Rightarrow g'(x) = -\frac{x-1}{x} < -1$$

let $f(x) = \ln(1+x) - x$ so that $f(0) = 0$.

Clearly

$$f'(x) = -\frac{x}{1+x}$$

and hence $g'(x) > 0$ if $-1 < x < 0$ and $f'(x) < 0$ if $x > 0$. It follows that that $f(x)$ in increasing in $(-1, 0]$ and decreasing in $[0, \infty)$. Thus we have $f(x) < f(0)$ if $-1 < x < 0$ and $f(x) < f(0)$ if $x > 0$. It thus follows that $f(x) \leq f(0) = 0$ for all $x > -1$ and there is equality only when $x = 0$. So we can write

$$\ln(1+x) \leq x \qquad \forall x \geq -1$$

Since x is just a dummy variable we can transform $x \to x - 1$ to get

$$\ln(x) \leq x - 1 \qquad \forall x \geq 0$$

This completes the proof. □

6.2.5. Prove that $\log_2 X = \frac{\log X}{\log 2}$. Interpret the meaning of

$$S = -\sum_i p_i \log_2(p_i)$$

.

Solution:

Let $y = \log_2 X$. Raising both sides to 2 a gives us

$$2^y = 2^{\log_2 x} \qquad \Rightarrow \qquad 2^y = x$$

Taking logarithm on both side with respect to base 110 we get

$$\log X = \log\left(2^y\right) \qquad \Rightarrow \log X = y \log 2 \qquad \Rightarrow \qquad y = \frac{\log X}{\log 2}$$

But by our assumption $y = \log_2 X$ thus we have

$$\log_2 X = \frac{\log X}{\log 2}$$

In digital electronics and in information theory where they represent the signal information in binary, the logarithm of a number with respect to 2 gives the total number of bits required to represent the number. Multiplying the number of bits $\log_2 N$ by the probability of the number gives the total average number of bits required.

So then the entropy function $S = -\sum_i p_i \log_2(p_i)$ represents the infromation content of the binary signal. □

6.3 Homework Three

6.3.1. Show that for a given N_r with $\sum_i^N p_i = 1$, the uncertainty function $S\left(\{p_i\}\right)$, takes its maximum value when $p_i = \frac{1}{N}$ for all i, that is $S\left(\{p_i\}\right) = A(N)$

Solution:

The uncertainty function is $S\left(\{p_i\}\right) = -C\sum_i p_i \ln p_i$. We want to maximize this function subject to the constraint $\sum_i p_i = 1$. Using Lagrange's multiplier method to find the extremum of function, we can define a new function $S - \lambda\left(\sum_i p_i - 1\right)$

$$\begin{aligned}
\frac{\partial S'}{\partial p_j} &= \frac{\partial}{\partial p_j}\left[-C\sum_i p_i \ln p_i - \lambda\left(\sum_i p_i - 1\right)\right] \\
&= -C\sum_i\left(\delta_{ij}\ln p_i + \frac{1}{p_j}p_i\delta_{ij}\right) - \lambda\left(\sum_i \delta_{ij}\right) \\
&= -C\left(\ln p_j + 1\right) - \lambda
\end{aligned} \tag{6.2}$$

But for extremum condition of this function the partial derivative with respect to every p_j should vanish. Thus we get

$$\ln p_j = -\frac{\lambda}{C} - 1 \qquad \Rightarrow \qquad p_j = \exp\left[-\frac{\lambda}{C} - 1\right]$$

The RHS of above expression is a constant, lets call that constant M so $p_i = M$ for some constant M but since probability has to add 1 we get

$$\sum_j p_j = 1; \qquad \Rightarrow \sum_j M = 1 \qquad \Rightarrow MN = 1 \qquad \Rightarrow M = \frac{1}{N}$$

Substuting this back we get

$$p_j = \frac{1}{M}$$

Thus the uncertainty function takes it maximum value when $p_i = 1/N$ for all p_i □

6.3.2. Consider a urn problem discussed in class: An urn is filled with balls, each numbered $n = 0, 1, 2 \ldots$. The average value of n is $\langle n \rangle = 2/7$. Calculate the probabilities p_0, p_1 and p_2 which yield the maximum uncertainty. Find the expectation value, based on these probabilities $\langle n^3 \rangle - 2\langle n \rangle$.
Solution:
The expectation value of n is given by

$$\langle n \rangle = p_0 \cdot 0 + p_1 \cdot 1 + p_2 \cdot 2 \quad \Rightarrow \quad p_1 + 2p_2 = 2/7$$

This is one of the constraints for maximizing the uncertainty function, the other constraint equation is $p_0 + p_1 + p_2 = 1$. Using these as we calculated in (6.2) we have

$$S' = \frac{S}{C} - \alpha(p_1 + 2p_2 - 2/7) - \beta(p_0 + p_1 + p_2 - 1)$$

Taking derivative with respect to α and β and equating to zero gives

$$\begin{aligned} \ln p_0 + 1 - \beta &= 0 \\ \ln p_1 + 1 - \alpha - \beta &= 0 \\ \ln p_2 + 1 - 2\alpha - \beta &= 0 \end{aligned}$$

These three equations along with two constraint equation form five equation in five unkonwn $p_0, p_1, p_2, \alpha, \beta$. We can solve this equation to get the numeric value of the parameters. Solving for the parameters we get

$$p_0 = \frac{15}{21} \qquad p_1 = \frac{4}{21} \qquad p_2 = \frac{1}{21}$$

Now the requried function is

$$\begin{aligned} \langle n^3 \rangle - 2\langle n \rangle &= p_0 \cdot 0 + p_1 \cdot 1^3 + p_2 \cdot 2^3 - \langle n \rangle \\ &= p_1 + 8p_2 - 2\frac{2}{7} \\ &= \frac{4}{21} + 8\frac{1}{21} - \frac{4}{7} \\ &= 0 \end{aligned}$$

The requried value is 0

□

6.3.3. Assuming the entropy, S and the number of microstates, Ω of a physical system are related through an arbitrary functional form $S = f(\Omega)$, show that the additive character of S (extensive parameter) and the multiplicative parameter Ω meaning $\Omega = \Omega_1, \Omega_2, \ldots$, is the number of microscopic states for a subsystem necessarily require that the function $F(\omega)$ is of the form

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Given the multiplicative parameter $\Omega = \Omega_1 \cdot \Omega_2 \ldots \Omega_r$. The extensive parameter as a function of this parameter which is a additive function be S. Thus we have

$$S(\Omega_1 \cdot \Omega_2 \ldots \Omega_r) = S(\Omega_1) + S(\Omega_1) + \ldots + S(\Omega_r)$$
$$S(\Omega) = \sum_j^r S(\Omega_j)$$

Differentiating with respect to Ω_i on both sides

$$\frac{\mathrm{d}}{\mathrm{d}\Omega_i} S(\Omega) = \frac{\mathrm{d}}{\mathrm{d}\Omega_i} \sum_j^r S(\Omega_j)$$
$$\frac{\mathrm{d}S(\Omega)}{\mathrm{d}\Omega} \frac{\mathrm{d}\Omega}{\mathrm{d}\Omega_i} = \sum_j^r \frac{\mathrm{d}S(\Omega_j)}{\mathrm{d}\Omega_i} \delta_{ij}$$

But since the derivative of product $\Omega = \Pi_j \Omega_j$ with respect to Ω_i is just the product without that parameter $\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega_i} = \Pi_{j \neq i} \Omega_j$. Multiplying both sides by Ω_i we get

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The expression in RHS is independent of expression on right. Since the product of the parameters can be varied while still keeping one of the parameters Ω_i constant. So the expression can only be equal to each other if they are equal to a constant.

$$\frac{\mathrm{d}S(\Omega)}{\mathrm{d}(\ln \Omega)} = k \qquad \Rightarrow \qquad dS(\Omega) = k d(\ln \Omega)$$

Integrating this expression we get

$$S(\Omega) = k \ln \Omega$$

Which is the required expression. □

6.3.4. Show that in $\ln x \leq x - 1$, if for all real positive x. The equality holds for $x = 1$.
Solution:
Rearranging the equation $\ln x - x \leq -1$. Let us define a function $g(x) = \ln x - x$. Differentiating this function with respect to x we get

$$g'(x) = \frac{1}{x} - 1 = \frac{1-x}{x} = -\frac{x-1}{x}$$

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let $f(x) = \ln(1+x) - x$ so that $f(0) = 0$.

Clearly

$$f'(x) = -\frac{x}{1+x}$$

and hence $g'(x) > 0$ if $-1 < x < 0$ and $f'(x) < 0$ if $x > 0$. It follows that that $f(x)$ in increasing in $(-1, 0]$ and decreasing in $[0, \infty)$. Thus we have $f(x) < f(0)$ if $-1 < x < 0$ and $f(x) < f(0)$ if $x > 0$. It thus follows that $f(x) \leq f(0) = 0$ for all $x > -1$ and there is equality only when $x = 0$. So we can write

$$\ln(1 + x) \leq x \qquad \forall x \geq -1$$

Since x is just a dummy variable we can transform $x \to x - 1$ to get

$$\ln(x) \leq x - 1 \qquad \forall x \geq 0$$

This completes the proof. □

6.3.5. Prove that $\log_2 X = \frac{\log X}{\log 2}$. Interpret the meaning of

$$S = -\sum_i p_i \log_2(p_i)$$

.

Solution:
Let $y = \log_2 X$. Raising both sides to 2 a gives us

$$2^y = 2^{\log_2 x} \qquad \Rightarrow \qquad 2^y = x$$

Taking logarithm on both side with respect to base 110 we get

$$\log X = \log\left(2^y\right) \qquad \Rightarrow \log X = y \log 2 \qquad \Rightarrow \qquad y = \frac{\log X}{\log 2}$$

But by our assumption $y = \log_2 X$ thus we have

$$\log_2 X = \frac{\log X}{\log 2}$$

In digital electronics and in information theory where they represent the signal information in binary, the logarithm of a number with respect to 2 gives the total number of bits required to represent the number. Multiplying the number of bits $\log_2 N$ by the probability of the number gives the total average number of bits required.

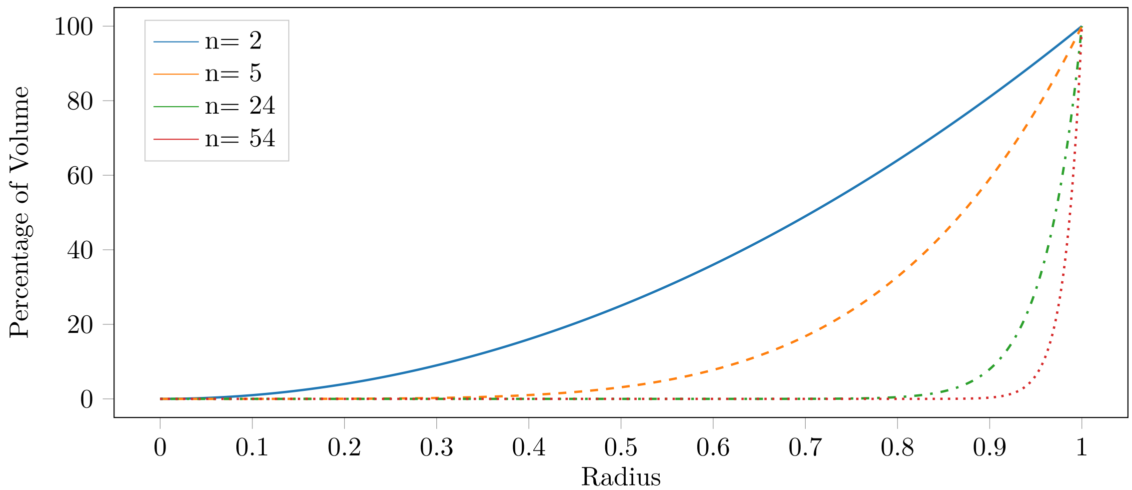
So then the entropy function $S = -\sum_i p_i \log_2(p_i)$ represents the infromation content of the binary signal. □

6.4 Homework Four

6.4.1. Consider an $N-$dimensional sphere.

(a) If a point is chosen at random in an $N-$ dimensional unit sphere, what is the probability of it falling inside the sphere of radius 0.99999999?
Solution:
The probability of a point falling inside a volume of radius r within a sphere of radius R is given by

$$p = \frac{V(r)}{V(R)} \tag{6.3}$$



where $V(x)$ is the volume of sphere of radius x. The volume of N dimensional sphere of radius x is

$$V(x) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} x^n$$

The progression of volume for different radius.
Using this in (6.3) we obtain

$$p = \left(\frac{r}{R}\right)^n \tag{6.4}$$

This gives the probability of a particle falling within a radius r in a Ndimensional sphere of radius R. □

(b) Evaluate your answer for $N = 3$ and $N = N_A$(the Avogadro Number)
Solution:
For $r = 0.999999$ and $N = 3$ and $N = N_A = 6.023 \times 10^{23}$ we get

$$p_3 = \left(\frac{0.999999}{1}\right)^3 = 0.999997000003 \qquad p_{N_A} = \left(\frac{0.999999}{1}\right)^{6.023\times10^{23}} = 0.0000000000000$$

The probability of a particle falling within the radius nearly 1 in higher two-dimensional sphere is vanishningly small. □

(c) What do these results say about the equivalence of the definitions of entropy in terms of either of the total phase space volume of the volume of outermost energy shell?
Solution:
Considering a phase space volume bounded by $E + \Delta$ where $\Delta \ll E$. The entropy of system bounded by the $E + \Delta$ and the outermost shell $\Sigma(E + \Delta) - \Sigma(E)$,

$$S_E = k \ln\left(\frac{\Sigma(E+\Delta)}{h^{3N}}\right), \qquad S_\Delta = k \ln\left(\frac{\Sigma(E+\Delta) - \Sigma(E)}{h^{3N}}\right)$$

Subtracting to see the differnce we get

$$S_E - S_\Delta = k \ln\left(1 - \frac{\Sigma(E)}{\Sigma(E+\Delta)}\right) \leq -\frac{\Sigma(E)}{\Sigma(E+\Delta)}$$

But for large dimension, the ration $\frac{\Sigma(E)}{\Sigma(E+\Delta)} \ll 0$. So we obtain

$$S_E - S_\Delta \approxeq 0 \qquad S_e \approxeq S_\Delta$$

This shows that the entropy interms of outrmost shell volume and the entire volume are almost the same. □

6.4.2. A harmonic oscillator has a Hamiltonian energy H related to its momentum P and its displacement q by the equation

$$p^2 + (M\omega q)^2 = 2MH$$

When $H = U$, a constant energy, sketch the path of the system in two-dimensional phase space.
Solution:
The phase space trajectory can be rearranged into

$$\frac{p^2}{\left(\sqrt{2MH}\right)^2} + \frac{q^2}{\left(\frac{1}{\omega}\sqrt{\frac{2H}{M}}\right)^2} = 1$$

This represents an ellipse in the phase space with semi major axis $a = \sqrt{2MH}$ and the semi minor axis $b = \frac{1}{\omega}\sqrt{\frac{2H}{M}}$.

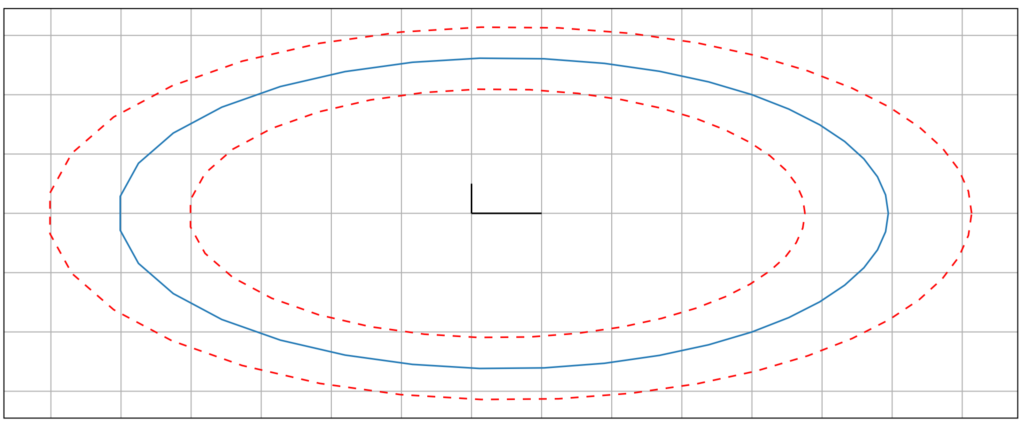


Figure 6.2: Phase plot of the system.

The volume of this 'volume' in phase space for constant energy $H = U$ is the area of ellipse which is

$$V = \pi ab = \pi\sqrt{2MU} \cdot \frac{1}{\omega}\sqrt{\frac{2U}{M}} = \frac{2\pi U}{\omega}$$

This gives the required phase space 'volume'. □
What volume of phase space does it enclose? In the case of N similar oscillators, which have the total energy U given by

$$\sum_{j=1}^{N} p^2 + \sum_{j=1}^{N}(M\omega q)^2 = 2MU$$

with additional coupling terms, too small to be included but large enough to ensure equipartition of energy, what is the nature of the path traversed by the system point?

(a) Show that the volume of the phase space "enclosed" by this path is $\frac{1}{N!}\left(\frac{2\pi U}{\omega}\right)^N$.
Solution:
Lets assume that the phase space volume of n harmonic oscillators which form a $2n$ dimensional ellipsoid be $C_n a^n b^n$ The coefficient can be found by usual method to be

$$C_n = \frac{\pi^n}{\Gamma(n+1)}$$

Noting that for this problem $a = \sqrt{2MU}$ and $b = \frac{1}{\omega}\sqrt{\frac{2U}{M}}$. The phase space volume becomes

$$\Sigma(U) = \frac{\pi^n}{\Gamma(n+1)} \left(\sqrt{2MU}\right)^n \left(\frac{1}{\omega}\sqrt{\frac{2U}{M}}\right)^n = \frac{1}{n!}\left(\frac{2\pi U}{\omega}\right)^n$$

This gives the required phase space volume. □

(b) Use the final result of (6.4.2a) to show that the entropy of N distinguishable harmonic oscillators, according to microcannonical ensemble is

$$S = Nk\left[1 + \ln\left(\frac{kT}{\hbar\omega}\right)\right]$$

Solution:
The entropy of system by definition is

$$S = k\ln\left(\frac{\Sigma(U)}{h^n}\right) = k\ln\left(\frac{1}{n!}\left(\frac{2\pi U}{h\omega}\right)^n\right) = k\ln\left(\frac{1}{n!}\right) + nk\ln\left(\frac{U}{\hbar\omega}\right)$$

Using Sterling's approximation for we get

$$\ln\left(\frac{1}{n!}\right) = -n\ln n + n$$

Susbtuting this back in the entropy equation gives

$$S = nk - nk\ln n + nk\ln\left(\frac{U}{\hbar\omega}\right) = nk\left[1 + \ln\left(\frac{U}{n\hbar\omega}\right)\right]$$

But for the simpile harmonic oscillator the energy $U = nkT$ using this gives

$$S = nk\left[1 + \ln\left(\frac{kT}{\hbar\omega}\right)\right]$$

This is the require expression for the entropy of the system. □

6.4.3. Consider a system fo N particles in which the energy of each particle can assume two and only two distinct values 0 and $E(>0)$. Denote by n_0 and n_1 the occupation numbers of energy level 0 and E, respectively. The total energy of the system is U.

(a) Find the entropy of such a system.
Solution:
Since there are $N = n_0 + n_1$ particles the total ways in which n_0 particle can go into 0 energy level is given by

$$\Omega = {}^N C_{n_0} = \frac{N!}{n_0! n_1!}$$

So the entropy of system is

$$S = k\ln\Omega = k\ln\left(\frac{N!}{n_0! n_1!}\right) = k\ln N! - k\ln n_0! - k\ln n_1!$$

For large N this can be simplified by using Sterling's approximation as

$$S = k(N\ln N - N + n_0 - n_0\ln n_0 + n_1 - n_1\ln n_1) = kN\left[\ln\left(\frac{N}{n_0}\right) + \frac{n_1}{N}\ln\left(\frac{n_0}{n_1}\right)\right]$$

This can be rearranged to obtain

$$S = -k\left[n_0 \ln\left(\frac{n_0}{N}\right) + n_1 \ln\left(\frac{n_1}{N}\right)\right]$$

This is the required entropy of the system. □

(b) Find the most probable value of the n_0 and n_1 and find the mean square fluctuations of these quantities.
Solution:
For this system, the energy constraint is

$$n_0 \cdot 0 + n_1 \cdot E = U$$

And the total number constraint is

$$N = n_0 + n_1$$

We have to maximize the function

$$\frac{S'}{k} = S - \alpha(n_0 + n_1 - N) + \beta\left(n_0 \cdot 0 + n_1 \cdot E - U\right)$$

Differentiating with respect to each occupation number n_0 and n_1 and α and β. We get

$$\ln n_1 + \alpha = 0$$
$$\ln n_1 + \alpha + E = 0$$

Solving these the only possible value of n_1 is

$$n_1 = \frac{U}{E} \qquad n_0 = N - n_1 = N - \frac{U}{E}$$

These are the possible values of n_0 and n_1 the occupation numbers.

□

(c) What happens when a system of negative temperature is allowed to exchange heat with a system of positive temperature?
Solution:
When the system of negative temperature is allowed to exchange energy with the system of positive energy the energy flows from the system of negative temperature to the system of positive temperature. □

6.4.4. **(Huang 6.4)** Using the corrected entropy formula, work out the entropy of mixing for the case of different gases for the case of identical gases, thus showing explicitly that there is no Gibbs paradox any more. Find also internal energy, U, and chemical potential, μ, using the corrected entropy formula and corrected entropy formula. The latter is called 'Sackur-Tetrode equation'.

Solution:
By using gibbs correction the phase space volume should be divided by $N!$

$$\Sigma(E) = \frac{1}{N!}\left(\frac{V}{h^3}\right)^N C_{3N} R^{3N} = \frac{1}{N!}\left(\frac{V}{h^3}\right)^N \left(\frac{2}{3N}\right)\frac{\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}\right)}(\sqrt{2ME})^{3N}$$

so the entropy function really becomes

$$S = k\ln\left(\Sigma(E)\right) = -N\ln N + N + Nk\ln\left[\frac{V}{h^3}R^3\right] + k\ln C_{3N}$$

Since N is a very large number we can make the better approximation of the Sterling approximation

$$\ln C_n = \ln\left(\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\right) \approx \frac{n}{2}\ln\left(\frac{2\pi e}{n}\right)$$

Which yields us

$$\begin{aligned} S &= k\left[\frac{3N}{2}\ln\left(\frac{2\pi e}{3N}\right) + N\ln\left(\frac{V}{Nh^3} + \frac{3}{2}N\ln(2mE)\right)\right] \\ &= Nk\left[\ln\left(\frac{4\pi mE}{3}\right)^{3/2} + \ln\left(\frac{V}{Nh^3}\right)\right] + \frac{3Nk}{2} \\ &= Nk\ln\left(\frac{V}{N}u^{3/2}\right) + \frac{3}{2}Nk\left[\frac{5}{3} + \ln\left(\frac{4\pi m}{3h^2}\right)\right] \end{aligned}$$

This is the fundamental equation of the system shich can be always inverted to find our intensive parameters. So the internal energy becomes

$$U = \left(\frac{3}{4}\frac{N\hbar^2}{\pi mV}\right)^{3/2}\exp\left(\frac{2S-5}{3Nk}\right)$$

□

6.5 Homework Five

6.5.1. (a) A system is composed of two harmonic osicllators, each of natural frequency ω_0 and each having permissible energies $\left(n+\frac{1}{2}\right)\hbar\omega_0$, where n is any non-negative integer. How many microstates are availabel to the system? What is the netropy of the system.

Solution:

Let the first oscillator be in n_1 and the second be in n_2 state. The total energy of the system then is the sum of the energies of each one

$$\left(n_1+\frac{1}{2}\right)\hbar\omega_0 + \left(n_2+\frac{1}{2}\right)\hbar\omega_0 = n'\hbar\omega_0 \qquad \Rightarrow n_1+n_2+1 = n'$$

The first of these oscillators can go to any one of n' states, but the second one is constrained to be in $n_2 = n' - n_1 - 1$ state. So there is freedom of only one choice among n' states. So the total number of microstates is just

$$\Omega = {}^{n'}C_1 = \frac{n'!}{1!(n'-1)!} = n'$$

So the entropy of the system is

$$S = k\ln(n')$$

In terms of energy of the system $E' = n'\hbar\omega_0$, the entropy becomes

$$S = k\ln\left(\frac{E'}{\hbar\omega_0}\right)$$

This is the required entropy of the system. □

(b) A second system is also composed of two harmonic oscillators, wach of natural frequencey $2\omega_0$. The total energy fo the system is $E'' = n''\hbar\omega_0$, where n'' is even integer. How many microstates are available in the system? What is the entropy of the system?
Solution:
Let the first oscillator be in n_1 and the second be in n_2 state. The total energy of the system then is the sum of the energies of each one

$$\left(n_1 + \frac{1}{2}\right) 2\hbar\omega_0 + \left(n_2 + \frac{1}{2}\right) 2\hbar\omega_0 = n''\hbar\omega_0 \qquad \Rightarrow n_1 + n_2 + 1 = n''/2$$

Since n'' is even integer the number $m = n''/2$ is another integer. The first of these oscillators can go to any one of m states, but the second one is constrained to be in $n_2 = m - n_1 - 1$ state. So the total number of microstates is just

$$\Omega = {}^mC_1 = \frac{m!}{1!(m-1)!} = m = \frac{n''}{2}$$

So the entropy of the system is

$$S = k \ln\left(\frac{n''}{2}\right)$$

In terms of energy of the system $E'' = n''\hbar\omega_0$, the entropy becomes

$$S = k \ln\left(\frac{E''}{2\hbar\omega_0}\right)$$

This is the required entropy of the system. □

(c) What is the entropy fo the sytem composed of te two preceeding sybsystems (separated and enclosed bya totally restrictive wall)? Express the entropy as a function of E'' and E'.
Solution:
The total entropy of the system is just the sum of individual entropies so

$$S = S_1 + S_2 = k \ln\left(\frac{E}{\hbar\omega_0}\right) + k \ln\left(\frac{E''}{2\hbar\omega_0}\right) = k \ln\left(\frac{E'E''}{2\hbar^2\omega_0^2}\right)$$

This gives the total entropy of the system composed of two given subsystems. □

6.5.2. A system consists of three distinguishable molecules at rest, each of which has a quantized magnetic moment, which can have its z-component $+M, 0$ and $-M$. Show that there are 27 different possible states of the system; list them all, giving the total $z-$component M_i of the magnetic moment for each. Compute the entropy $S = -k\sum_i f_i \ln f_i$ of the system for the following priori porbabilities:

A:	-M	-M	-M	-M	-M	-M	-M	-M	-M	M	M	M	M	M	M	M	M	M	0	0	0	0	0	0	0	0	0
B:	-M	-M	-M	M	M	M	0	0	0	-M	-M	-M	M	M	M	0	0	0	-M	-M	-M	M	M	M	0	0	0
C:	-M	M	0	-M	M	0	-M	M	0	-M	M	0	-M	M	0	-M	M	0	-M	M	0	-M	M	0	-M	M	0
Sum:	-3M	-M	-2M	-M	M	0	-2M	0	-M	-M	M	0	M	3M	2M	0	2M	M	-2M	0	-M	0	2M	M	-M	M	0

(a) All 27 states are equally likely.
Solution:
If all states are equally likely then the probability of each state is $f_i = \frac{1}{27}$. So the total entropy of system is

$$S = -k \sum_i f_i \ln f_i = \sum_{i=1}^{27} \frac{1}{27} \ln\left(\frac{1}{27}\right) = k \ln 27$$

□

(b) Each state is equally likely for which the z-component M_z of the total magnetic moment is zero; $f_i = 0$ for all other states.
Solution:
There are six states where the total moment is zero. So the if all them are equally likely and the rest has probability $f_i = 0$ then we have

$$S = -k \sum_i f_i \ln f_i = \sum_{i=1}^{6} \frac{1}{6} \ln\left(\frac{1}{6}\right) = k \ln 6$$

This gives the required entropy. □

(c) Each state is equally likely for which $M_z = M$; $f_i = 0$ fro all other states.
Solution:
There are seven states where the total moment is M. So the if all them are equally likely and the rest has probability $f_i = 0$ then we have

$$S = -k \sum_i f_i \ln f_i = \sum_{i=1}^{7} \frac{1}{7} \ln\left(\frac{1}{7}\right) = k \ln 7$$

This gives the required entropy. □

(d) Each state is equally likely for which $M_2 = 3M$ $f_i = 0$ for all other states.
Solution:
There is just one state where the total moment is 3M. So the if all them are equally likely and the rest has probability $f_i = 0$ then we have

$$S = -k \sum_i f_i \ln f_i = \sum_{i=1}^{1} \frac{1}{1} \ln\left(\frac{1}{1}\right) = k \ln 1 = 0$$

This gives the required entropy. □

(e) The distribution for which S is maximum subject to the requirement that $\sum f_i = 1$ and the mean component $\sum_i f_i M_i = \gamma M$. Show that for this distribution

$$f_i = \frac{e^{(3M - M_i)\alpha}}{(1 + x + x^2)^3}$$

where $x = e^{\alpha M}$(α begin Lagrange multiplier) and where the value fo x is determined by equation $\gamma = \frac{3(1-x^)2}{1+x+x^2}$. Compute x and S for $\gamma = 3$ and compare your answers.
Solution:
The entropy of the system is $S = -k \sum f_i \ln f_i$. We have to maximize this function subject to the constraints $\sum_i f_i = 1$ and $\sum_i f_i M_i = \gamma M$. Using lagranges multiplier technique the function to maximize the function $\frac{S}{k}$ is

$$F = \sum_i f_i \ln f_i - \alpha' \left(\sum_i f_i - 1\right) - \beta \left(\sum_i f_i M_i - \gamma M\right)$$

Differentiating with respect to f_j and setting equal to 0 we get

$$\begin{aligned}\frac{\partial F}{\partial f_j} &= \sum_i \left[\frac{\partial f_i}{\partial f_j}\ln f_i + f_i\frac{\partial \ln f_i}{\partial f_j}\right] - \alpha'\sum_i \frac{\partial f_i}{\partial f_j} - \beta\sum M_i\frac{\partial f_i}{\partial f_j}\\ &= \sum_i\left[\delta_{ij}\ln f_i + f_i\frac{1}{f_j}\delta_{ij}\right] - \alpha'\sum_i\delta_{ij} - \beta\sum_i M_i\delta_{ij}\\ &= \ln f_j + \underbrace{1-\alpha'}_{\alpha} - \beta M_j\end{aligned}$$

For maximum the derivative has to vanish, setting this derivative equal to zero we have

$$\ln f_i + \alpha - \beta M_j = 0 \qquad \Rightarrow f_i = e^{-\alpha+\beta M_i} \tag{6.5}$$

The sum of probability constraint and the average constraint are

$$\sum_i f_i = \sum_i e^{-\alpha+\beta M_i} = 1 \qquad \Rightarrow \qquad e^{\alpha} = \sum_i e^{\beta M_i}$$

The last expression on the right can be written as the sum over all the total moments M_i with multiplicity $g(M_i)$ as

$$e^{\alpha} = \sum_{M_i} g(M_i)e^{\beta M_i}$$

Looking at the configuration table we have that the multiplicity for each states is

$$g(3M) = g(-3M) = 1 \qquad g(-2M) = g(2M) = 3 \qquad g(-M) = g(M) = 6 \qquad g(0) = 7$$

Denoting $x = e^{\beta M}$ we have

$$\begin{aligned}e^{\alpha} &= g(-3M)x^{-3} + g(-2M)x^{-2} + g(-M)x^{-1} + g(0)x^0 + g(M)x + g(2M)x^2 + g(3M)x^3\\ &= x^{-3} + 3x^{-2} + 6x^{-1} + 7 + 6x + 3x^2 + x^3\\ &= x^{-3}\left(1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6\right)\\ &= \frac{\left(1+x+x^2\right)^3}{x^3}\end{aligned} \tag{6.6}$$

Substuting this back into (6.5) we get

$$f_i = \frac{x^3 e^{\beta M_i}}{\left(1+x+x^2\right)^3} = \frac{e^{3\beta M + \beta M_i}}{\left(1+x+x^2\right)^3} = \frac{e^{(3M+M_i)\beta}}{\left(1+x+x^2\right)^3}$$

Now invoking the average moment constraint we get

$$\sum_i f_i M_i = \gamma M \qquad \Rightarrow \sum_i \frac{x^3 e^{(M_i)\beta}}{\left(1+x+x^2\right)^3} M_i = \gamma M$$

Using $e^{\alpha} = \sum_i e^{\beta M_i}$ the expression becomes

$$\frac{x^3}{\left(1+x+x^2\right)^3}\frac{\partial}{\partial\beta}\left[\sum_i e^{\beta M_i}\right] = \gamma M \qquad \Rightarrow \qquad \frac{x^3}{\left(1+x+x^2\right)^3}\frac{\partial e^{\alpha}}{\partial \beta} = \gamma M$$

Since we have $x = e^{\beta M}$, this can be differentiated to get, $\frac{\mathrm{d}x}{\mathrm{d}\beta} = Mx$. And substuting e^{α} from (6.6) the above expression becomes

$$\begin{aligned}\frac{x^3}{\left(1+x+x^2\right)^3}\frac{\partial}{\partial x}\left[\frac{\left(1+x+x^2\right)^3}{x^3}\right]\frac{\partial x}{\partial \beta} &= \gamma M\\ \frac{x^3}{\left(1+x+x^2\right)^3}\left[\frac{3(x^2-1)\left(1+x+x^2\right)^2}{x^4}\right] Mx &= \gamma M\\ \frac{3(x^2-1)}{1+x+x^2} &= \gamma\end{aligned}$$

Solving this equation for various values of γ we get a quadratic equation.

$$(\gamma-3)x^2+\gamma x+(\gamma+3)=0 \qquad \Rightarrow x=\frac{\gamma\pm\sqrt{3(12-\gamma^2)}}{2(3-\gamma)}$$

Specifically for $3\geq\gamma\geq 0$ we have to choose the positive sign so

$$x=\frac{\gamma+\sqrt{3(12-\gamma^2)}}{2(3-\gamma)}$$

For the various values computed are

γ	x	S
0.0	1.0	-3.29584
1.0	1.69	-3.04037
3.0	-2.0	∞

This gives the various values of entropy for the given values of γ. □

6.5.3. Prove that for a system in cannonical ensemble

$$\left\langle \Delta E^3\right\rangle = k^2\left[T^4\left(\frac{\partial C_v}{\partial T}\right)_V+2T^3C_v\right]$$

in particular, for ideal gas

$$\left\langle\left(\frac{\Delta E}{U}\right)^2\right\rangle=\frac{2}{3N} \qquad \text{and} \left\langle\left(\frac{\Delta E}{U}\right)^3\right\rangle=\frac{8}{9N^2}$$

Solution:
The expectation value of cube of fluctation of E from mean value can be written as

$$\begin{aligned}\left\langle \Delta E^3\right\rangle = \left\langle (E-\langle E\rangle)^3\right\rangle = \left\langle E^3-3E^2\langle E\rangle+3E\langle E\rangle^2-\langle E\rangle^3\right\rangle \\ = \left\langle E^2\right\rangle-3\left\langle E^2\right\rangle\langle E\rangle+3\langle E\rangle\left\langle E^2\right\rangle-\langle E\rangle^3 \\ \left\langle \Delta E^3\right\rangle = \left\langle E^2\right\rangle-3\left\langle E^2\right\rangle\langle E\rangle+2\langle E\rangle^3\end{aligned} \tag{6.7}$$

In light of (6.7) The average energy of the system can be written as

$$U=\langle E\rangle=\frac{\sum E_ie^{-\beta E_i}}{\sum e^{-\beta E_i}} \tag{6.8}$$

Differentiating (6.7) with respect to β we get

$$\begin{aligned}\frac{\partial U}{\partial\beta} &= -\frac{\sum E_i^2e^{-\beta E_i}}{\sum e^{-\beta E_i}}+\left[-\sum E_ie^{-\beta E_i}\frac{\sum E_ie^{-\beta E_i}}{-\left(\sum e^{-\beta E_i}\right)^2}\right] \\ &= -\frac{\sum E_i^2e^{-\beta E_i}}{\sum e^{-\beta E_i}}+\left[\frac{\sum E_ie^{-\beta E_i}}{\sum e^{-\beta E_i}}\right]^2 \\ &= -\frac{\sum E_i^2e^{-\beta E_i}}{\sum e^{-\beta E_i}}+U^2\equiv\left\langle E^2\right\rangle-\langle E\rangle^2\end{aligned} \tag{6.9}$$

Differentiating (6.9) again with respect to β we get

$$\begin{aligned}\frac{\partial^2 U}{\partial \beta^2} &= \frac{\sum E_i^3 e^{-\beta E_i}}{\sum e^{-\beta E_i}} - \left[\frac{\sum E_i^2 e^{-\beta E_i}}{-\left(\sum e^{-\beta E_i}\right)^2} \cdot \sum -E_i e^{-\beta E_i}\right] + 2U\frac{\partial U}{\partial \beta} \\ &= \frac{\sum E_i^3 e^{-\beta E_i}}{\sum e^{-\beta E_i}} - \left[\frac{\sum E_i^2 e^{-\beta E_i}}{\sum e^{-\beta E_i}} \cdot \frac{\sum E_i e^{-\beta E_i}}{\sum e^{-\beta E_i}}\right] + 2\left\langle E\right\rangle\left[-\left\langle E^2\right\rangle + \left\langle E\right\rangle^2\right] \\ &= \left\langle E^3\right\rangle - \left[\left\langle E^2\right\rangle\left\langle E\right\rangle\right] - 2\left\langle E\right\rangle\left\langle E^2\right\rangle + 2\left\langle E\right\rangle^3 \\ &= \left\langle E^3\right\rangle - 3\left\langle E^2\right\rangle\left\langle E\right\rangle + 2\left\langle E\right\rangle^3 \end{aligned} \tag{6.10}$$

Comparing (6.7) and (6.10) we get

$$\left\langle \Delta E^3\right\rangle = \frac{\partial^2 U}{\partial \beta^2} = \frac{\partial}{\partial \beta}\left(\frac{\partial U}{\partial T}\frac{\partial T}{\partial \beta}\right) = \frac{\partial}{\partial T}\left(\frac{\partial U}{\partial T}\frac{\partial T}{\partial \beta}\right)\frac{\partial T}{\partial \beta}$$

Since $\beta = \frac{1}{kT}$ the derivative $\frac{\partial T}{\partial \beta} = -kT^2$ and recocniging that $\frac{\partial U}{\partial T} = C_v$ we get

$$\left\langle \Delta E^3\right\rangle = \frac{\partial}{\partial T}\left(-kT^2 C_v\right)\left(-kT^2\right) = \left(kT^2\frac{\partial C_v}{\partial T} + 2kTC_v\right)\left(kT^2\right) = k^2T^3\left(T\frac{\partial C_v}{\partial T} + 2C_v\right)$$

This is the required expression for $\left\langle \Delta E^3\right\rangle$. Using $U = 3NkT; U^2 = 9N^2K^2T^2$ and with $C_v = 3NK$, $\frac{\partial C_v}{\partial T} = 0$ and substutig back in the expression we get

$$\left\langle\left(\frac{\Delta E}{U}\right)^2\right\rangle = \frac{2}{3N} \qquad \text{and} \left\langle\left(\frac{\Delta E}{U}\right)^3\right\rangle = \frac{8}{9N^2}$$

These are the required values for ideal gas. □

6.5.4. Verify that, for ideal gas,

$$\frac{S}{Nk} = \ln\left(\frac{Q_1}{N}\right) + T\left(\frac{\partial \ln Q_1}{\partial T}\right)_P$$

Solution:
For an ideal gas, we assume that each molecule is free and so they dont exert force on each other, so the potential is zero. Also they have same momemtum in all directions which leads to the hamiltonian

$$H = \sum_i \frac{p_i^2}{2m} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

The partition function for a single molecule is

$$Q_1 = \frac{1}{h^3}\int e^{-\beta H} d^3q d^3p = \frac{1}{h^3}\left[\int_V d^3q\right] \cdot \left[\int_{-\infty}^{\infty} e^{-\beta H} d^3p\right]$$

The integration of the space coordinates q_i just makes gives the volume of the system as it is independent of the momentum coordinates

$$Q_1 = \frac{V}{h^3}\int_{\infty}^{\infty} \exp\left[-\frac{1}{\beta}\left(\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right)\right] dp_x dp_y dp_z$$

Since the momentum in each direction can be considered to be the same and the parameter $\beta = \frac{1}{kT}$ we get

$$Q_1 = \frac{V}{h^3}\int_{\infty}^{\infty} \exp\left[-\frac{1}{kT}\left(\frac{p^2}{2m} + \frac{p^2}{2m} + \frac{p^2}{2m}\right)\right] dpdpdp = \frac{V}{h^3}\left[2\int_0^{\infty} \exp\left(-\frac{p^2}{2mkT}\right) dp\right]^3$$

This integral is just the gamma function and the integral is easily computed to be $\sqrt{\frac{mkT}{2}} \cdot \sqrt{\pi}$. So the partitiion function becomes

$$Q_1 = \frac{V}{h^3}\left[\sqrt{2mkT} \cdot \sqrt{\pi}\right]^3 = V\left(\frac{2\pi mkT}{h^2}\right)^{3/2}$$

Also for ideal gas the relation $PV = NKT$ takig the various derivatives of the partition function we get

$$\ln\left(\frac{Q_1}{N}\right) = \ln\left[\frac{KT}{P}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right]$$
$$\left(\frac{\partial \ln Q_1}{\partial T}\right)_P = \frac{\partial}{\partial T}\left[\ln(NKT) - \ln P + \frac{3}{2}\ln\left(\frac{2\pi mkT}{h^2}\right)\right]_P = \left[\frac{1}{T} - 0 + \frac{3}{2}\frac{1}{T}\right] = \frac{5}{2T}$$

Combining these two we get

$$\ln\left(\frac{Q_1}{N}\right) + T\left(\frac{\partial \ln Q_1}{\partial T}\right) = \ln\left[\frac{kT}{P}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right] + \frac{5}{2}$$

The expression on the right is just $\frac{S}{Nk}$ □

Chapter 7

Quantum Mechanics II

7.1 Homework One

7.1.1. **(Sakurai 2.33)** The propagator in momentum space is given by $\langle \boldsymbol{p''}, t | \boldsymbol{p'}, t_0 \rangle$. Derive an explicit expression for $\langle \boldsymbol{p''}, t | \boldsymbol{p'}, t_0 \rangle$ for the free particle case.
Solution:
For a free particle the Hamiltonian is

$$H = \frac{\boldsymbol{p}^2}{2m}$$

So the time evolution operator for any state in momentum space is given by

$$\mathcal{U}(t) = e^{\frac{iHt}{\hbar}} \qquad \Rightarrow \qquad \exp\left[\frac{i\boldsymbol{p}^2 t}{2m\hbar}\right]$$

The base kets evolve over time as

$$|\boldsymbol{p'}, t\rangle = \mathcal{U}(t)^\dagger \, |\boldsymbol{p'}, 0\rangle \qquad \rightarrow \langle \boldsymbol{p'}, t| = \langle \boldsymbol{p'}, 0| \, \mathcal{U}(t) = \langle \boldsymbol{p'}, t_0| \exp\left[\frac{i\boldsymbol{p}^2 t}{2m\hbar}\right]$$

So the propagator becomes

$$\begin{aligned}
\langle \boldsymbol{p''}, t | \boldsymbol{p'}, t_0 \rangle &= \langle \boldsymbol{p''}, 0 | \exp\left[\frac{i{p''}^2 t}{2m\hbar}\right] \exp\left[-\frac{i{p'}^2 t_0}{2m\hbar}\right] |\boldsymbol{p'}, 0\rangle \\
&= \exp\left[\frac{i}{2m\hbar}\left(p''^2 t - p'^2 t_0\right)\right] \langle \boldsymbol{p''}, 0 | \boldsymbol{p'}, 0\rangle \\
&= \exp\left[\frac{i}{2m\hbar}\left(p''^2 t - p'^2 t_0\right)\right] \delta(\boldsymbol{p''} - \boldsymbol{p'})
\end{aligned}$$

This gives explicit expression for the propagator of the free particle. □

7.1.2. **(Skurai 2.37)**

(a) Verify $[\Pi_i, \Pi_j] = \left(\frac{i\hbar e}{c}\right) \varepsilon_{ijk} B_k$. and $m \frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\Pi}{\mathrm{d}t} = e\left[\boldsymbol{E} + \frac{1}{2c}\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \times \boldsymbol{B} - \boldsymbol{B} \times \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)\right]$
Solution:

The kinematical momentum for electromagnetic field is defined as $\boldsymbol{\Pi} \equiv m\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{p} - \frac{e\boldsymbol{A}}{c}$ where $\boldsymbol{A}$ is the vector magnetic potential is a function of operator $\boldsymbol{x}$. The commutator then is

$$\begin{aligned}
[\Pi_i, \Pi_j] &= \left[p_i - \frac{e}{c}A_i, p_j - \frac{e}{c}A_j\right] \\
&= [p_i, p_j] - \left[p_i, \frac{e}{c}A_j\right] - \left[\frac{e}{c}A_i, p_j\right] + \left[\frac{e}{c}A_i, \frac{e}{c}A_j\right] \\
&= 0 - \frac{e}{c}\left(-i\hbar\frac{\partial A_j}{\partial x_i}\right) - \frac{e}{c}\left(i\hbar\frac{\partial A_i}{\partial x_j}\right) + 0 \\
&= \frac{i\hbar e}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) \\
&= \frac{i\hbar e}{c}B_k
\end{aligned}$$

repeating this same process for all the components of this kinematical momentum operator we get

$$[\Pi_i, \Pi_j] = \frac{i\hbar e}{c}\varepsilon_{ijk}B_k \tag{7.1}$$

The Hamiltonian for electromagnetic field id $H = \frac{\Pi^2}{2m} + e\phi$. For the Lorentz force formula we have $m\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \equiv \boldsymbol{\Pi}$ differentiating this with time gives $m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\boldsymbol{\Pi}}{\mathrm{d}t}$ by using Heisenberg equation of motion we can write

$$\begin{aligned}
m\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = \frac{\mathrm{d}\Pi_i}{\mathrm{d}t} &= \frac{1}{i\hbar}[\Pi_i, H] \\
&= \frac{1}{i\hbar}\left[\Pi_i, \frac{\Pi^2}{2m} + e\phi\right] \\
&= \frac{1}{i\hbar}\left[\Pi_i, \frac{\boldsymbol{\Pi}^2}{2m}\right] + \frac{1}{i\hbar}\left[p_i + \frac{e}{c}A_x, e\phi\right] \\
&= \frac{1}{2mi\hbar}\sum_j \left[\Pi_i, \Pi_j^2\right] + \frac{1}{i\hbar}[p_i, e\phi]
\end{aligned}$$

But the commutator of $\left[\Pi_i, \Pi_j^2\right] = \Pi_r[\Pi_i, \Pi_j] + [\Pi_i, \Pi_j]\Pi_r$ which by use of (7.1) reduces to

$$\left[\Pi_i, \Pi_j^2\right] = \Pi_j\frac{i\hbar e}{c}\varepsilon_{ijk}B_k + \frac{i\hbar e}{c}\varepsilon_{ijk}B_k\Pi_j$$

And also $\frac{1}{i\hbar}[p_i, e\phi] = \frac{1}{i\hbar}(-i\hbar)\frac{\partial e\phi}{\partial x} = -eE_i$
Using these two facts back in in the original commutator leads to

$$\begin{aligned}
m\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} &= \frac{1}{2mi\hbar}\sum_j \varepsilon_{ijk}p_jB_k\frac{i\hbar e}{c} + \varepsilon_{ijk}B_k\frac{i\hbar e}{c}p_j - eE_i \\
&= e\left[E + \frac{1}{2c}\sum_j\left(\frac{\mathrm{d}x_j}{\mathrm{d}t}B_k - B_j\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)\right]
\end{aligned}$$

The above expression can be obtained for each components ij and k to obtain the required relation in 3D

$$m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\Pi}{\mathrm{d}t} = e\left[\boldsymbol{E} + \frac{1}{2c}\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \times \boldsymbol{B} - \boldsymbol{B} \times \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)\right]$$

This is the required lorentz force relation. □

(b) Verify $\frac{\partial\rho}{\partial t} + \nabla' \cdot \boldsymbol{j} = 0$ with $\boldsymbol{j}$ given by $\boldsymbol{j} = \left(\frac{\hbar}{m}\operatorname{Im}(\psi^*\nabla'\psi) - \left(\frac{e}{mc}\right)\boldsymbol{A}|\psi|^2,\right)$

Solution:

By definition the probability density function is the absolute value square of waefunction. The Hamiltonian for electromagnetic field for arbitrary wavefunction ψ is given by

$$H = \frac{\Pi^2}{2m} + e\phi = \frac{2}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + e\phi$$

The momentum operator in position space wavefunction can be written as $-i\hbar\nabla$. Using the schrodinger equation $H\psi = E\psi$ where operator E is given by $E = i\hbar\frac{\partial}{\partial t}$ we get

$$\begin{aligned}
H\psi &= i\hbar\frac{\partial}{\partial t}\psi \\
\frac{\partial\psi}{\partial t} &= \frac{1}{i\hbar}\left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)^2 + e\phi\right] \\
&= \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2 + i\hbar\frac{e}{2mc}\left(\nabla\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\nabla\right) + \frac{e^2}{2mc^2}A^2 + e\phi\right]\psi \\
&= \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2\psi + i\hbar\frac{e}{2mc}\left(\nabla\cdot(\boldsymbol{A}\psi) + \boldsymbol{A}\cdot\nabla\psi\right) + \frac{e^2}{2mc^2}A^2\psi + (e\phi)\psi\right] \\
&= \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{2mc}\left(\nabla\cdot\boldsymbol{A}\right)\psi + \frac{e}{2mc}\boldsymbol{A}\cdot\nabla\psi + \frac{e}{2mc}\boldsymbol{A}\cdot\nabla\psi + \frac{-i}{\hbar}\left(\frac{e^2}{2mc^2}A^2 + e\phi\right)\psi \\
&= \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{2mc}\left(\nabla\cdot\boldsymbol{A}\right)\psi + \frac{e}{mc}\boldsymbol{A}\cdot\nabla\psi + \frac{-i}{\hbar}\left(\frac{e^2}{2mc^2}A^2 + e\phi\right)\psi
\end{aligned}$$

Taking the conjugate of this expression leads to

$$\frac{\partial\psi^*}{\partial t} = \frac{-i\hbar}{2m}\nabla^2\psi^* + \frac{e}{2mc}\left(\nabla\cdot A\right)\psi^* + i\hbar\frac{e}{mc}\boldsymbol{A}\cdot\nabla\psi + \frac{i}{\hbar}\left(\frac{e^2}{2mc^2}A^2 + e\phi\right)\psi^* \tag{7.2}$$

Taking the time derivative of the probability density function we get

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t} = \frac{\partial}{\partial t} = \psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi$$

For a divergence free magnetic vector potential (which we can always choose), Multiplying (7.2) by ψ and its conjugate by ψ^* and adding we get

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \psi^*\frac{i\hbar}{2m}\nabla^2\psi + \psi^*\frac{e}{mc}\boldsymbol{A}\cdot(\nabla\psi) + \psi\frac{-i\hbar}{2m}\nabla^2\psi^* + \psi\frac{e}{mc}\boldsymbol{A}\cdot(\nabla\psi^*) \\
&= \frac{i\hbar}{2m}\left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right] + \frac{e}{mc}\left(\psi\boldsymbol{A}\cdot(\nabla\psi^*) + \psi^*\boldsymbol{A}\cdot(\nabla\psi)\right) \\
&= \frac{i\hbar}{2m}(2i\nabla\cdot\operatorname{Im}(\psi^*\nabla\psi) + \frac{e}{mc}(\nabla\cdot(\boldsymbol{A}\psi^*\psi)) \\
&= -\frac{\hbar}{m}\nabla\cdot(\operatorname{Im}(\psi^*\nabla\psi)) + \frac{e}{mc}\nabla\cdot(\boldsymbol{A}|\psi|^2) \\
&= -\nabla\cdot\left(\frac{\hbar}{m}\operatorname{Im}(\psi^*\nabla\psi) - \frac{e}{mc}\boldsymbol{A}|\psi|^2\right) \\
&= -\nabla\cdot\boldsymbol{j}
\end{aligned}$$

This completes the proof. □

7.1.3. **(Sakurai 2.38)** Consider a Hamiltonian of the spinless particle of charge e. In presence of a static magnetic field, the interaction terms can be generated by

$$\boldsymbol{P}_{\text{operator}} \to \boldsymbol{P}_{\text{operator}} - \frac{e\boldsymbol{A}}{c},$$

where $\boldsymbol{A}$ is the appropriate vector potential. Suppose, for simplicity, the magnetic field $\boldsymbol{B}$ is uniform in the positive $z-$ direction. Prove that the above presciprition indeed leads to the correct expression for the interaction of the orbital magnetic moment $(e/2mc)\,\boldsymbol{L}$ with the magnetic field $\boldsymbol{B}$. Show that there is also an extra term proportional to $B^2(x^2+y^2)$, and comment briefly on its physical significance.

Solution:
Since the electric field is zero we can assign a scalar potential as constant and the constant can always be chosen 0 thus $\phi = 0$. The vector magnetic potential for uniform magnetic field is $\boldsymbol{A} = \frac{1}{2}\boldsymbol{x} \times B\hat{\mathbf{z}}$. Since there is a free choice of vector magnetic potential as long as its curl is divergence free, we chose this potential which is also divergence free. Thus for this case $\nabla \cdot A = 0$.

From (7.2) we have the hamiltonian of the system is

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar e}{mc}\boldsymbol{A}\cdot\nabla + \frac{i\hbar e}{2mc}\nabla\cdot\boldsymbol{A} + \frac{e^2}{2mc^2}\boldsymbol{A}^2$$

Since $\nabla\cdot\boldsymbol{A} = 0$ by our choice the interaction operator terms introduced due to the presence of magnetic potential is

$$\frac{i\hbar e}{mc}\boldsymbol{A}\cdot\nabla + \frac{e^2}{2mc^2}\boldsymbol{A}^2 = -\frac{e}{mc}\boldsymbol{A}\cdot(-i\hbar\nabla) + \frac{e^2}{2mc^2}\boldsymbol{A}^2$$

But the operator $-i\hbar\nabla$ is the momentum operator $\boldsymbol{p}$ and $\boldsymbol{A}^2 = \frac{1}{4}B^2\left(x^2+y^2\right)$ This enables us to write the interaction terms as

$$-\frac{e}{mc}B\frac{1}{2}\left(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\right)\cdot\boldsymbol{p} + \frac{e^2B^2}{8mc^2}\left(x^2+y^2\right)$$

We can recocnize the term $(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})\cdot\boldsymbol{p} = -yP_x + xP_y = L_z$ Substuting this in the above expression we get

$$\frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}\left(x^2+y^2\right)$$

So the final hamiltonian becomes

$$H = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}\left(x^2+y^2\right)\right]$$

So the interaction terms introduced in the absence of scalar potential but the presence of magnetic potential has operator for orbital angular momentum $\frac{e}{2mc}BL_z$ and a term proportional to $B^2\left(x^2+y^2\right)$ □

7.1.4. **(Sakurai 2.39)** An electron moves in he presence of a uniform magnetic field in the z-direction ($\boldsymbol{B} = B\hat{\mathbf{z}}$)

(a) Evaluate $[\Pi_x, \Pi_y]$ where $\Pi_x \equiv p_x - \dfrac{eA_x}{c}, \qquad \Pi_y \equiv p_y - \dfrac{eA_y}{c}$.

Solution:

$$\begin{aligned}
[\Pi_x, \Pi_y] &= \left[p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y\right] \\
&= [p_x, p_y] - \left[p_x, \frac{e}{c}A_y\right] - \left[\frac{e}{c}A_i, p_j\right] + \left[\frac{e}{c}A_i, \frac{e}{c}A_j\right] \\
&= 0 - \frac{e}{c}\left(-i\hbar\frac{\partial A_j}{\partial x}\right) - \frac{e}{c}\left(i\hbar\frac{\partial A_x}{\partial x_y}\right) + 0 \\
&= \frac{i\hbar e}{c}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{i\hbar e}{c}B_z
\end{aligned}$$

Which is the required expression for the comutator. □

(b) By comparing the Hamiltonian and the commutation relation obtained in 7.1.4 with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2 k^3}{2m} + \left(\frac{|eB|\hbar}{mc}\right)\left(n + \frac{1}{2}\right)$$

Solution:
Since the charged particle is only in the magnetic field, the electric field is absent, which means the electric potential is a constant whiich we may assume to be 0. So the hamiltonian of the system is

$$H = \frac{\mathbf{\Pi}^2}{2m} = \frac{\Pi_z^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{\Pi_x^2}{2m}$$

The energy eigenvalue equation for the a general wavefunction $\psi_\alpha(x')$ we have

$$H\psi_\alpha(x') = \left[\frac{\Pi_z^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{\Pi_x^2}{2m}\right]\psi_\alpha(x')$$

Since the magnetic field is completely in $\hat{\mathbf{z}}$ the vector magnetic potential can be written as $A(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x} \times B\hat{\mathbf{z}}$ so that $A_z = 0$. This simplifies the eigenvalue equation to

$$H\psi_\alpha(x') = \left[\frac{p_z^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{\Pi_x^2}{2m}\right]\psi_\alpha(x')$$

The first of these three expression p_z has known eigenvalue $\hbar k$ given in the problem. The second two terms can be evaluated using the One dimensional simple harmonic oscillator. Since the comutator $[\Pi_x, \Pi_y] = i\hbar\frac{e}{c}B$ we can scale Π_y by $\frac{c}{eB}$ to make $\left[\Pi_x, \frac{c}{eB}\Pi_y\right] = i\hbar$. Let $Y = \frac{c}{eB}\Pi_y$ Using this the expression becomes

$$H\psi_\alpha(x') = \left[\frac{p_z^2}{2m} + \frac{\Pi_x^2}{2m} + \frac{1}{2}m\frac{e^2B^2}{m^2c^2}Y^2\right]\psi_\alpha(x')$$

We can again try the raising a operator and lowering operators $a^\dagger$ out of the last two expression.

$$a = \sqrt{\frac{eB}{2\hbar c}}\left(Y + \frac{ic}{eB}\Pi_x\right) \qquad a^\dagger = \sqrt{\frac{eB}{2\hbar c}}\left(Y - \frac{ic}{eB}\Pi_x\right)$$

And since $a^\dagger a = \frac{mc}{\hbar eB}H + \frac{i}{2\hbar}[Y, \Pi_x] = \frac{Hmc}{\hbar eB} - \frac{1}{2}$. In complete analogy to SHO we find $a^\dagger a$ works as simultaneous operator with Hamiltonian H, i.e. $a^\dagger a$ commutes with H, and so acts on energy eigenstates to give integer n as its eigenvalue. So the eigenvalue become

$$H\psi_\alpha(x') = \left[\frac{p_z^2}{2m}\psi_\alpha(x')\right] + \left[\frac{\Pi_x^2}{2m} + \frac{1}{2}m\frac{e^2B^2}{m^2c^2}Y^2\right]\psi_\alpha(x')$$
$$H\psi_\alpha(x') = \frac{\hbar^2k^2}{2m}\psi_\alpha(x') + \left[\left(n + \frac{1}{2}\right)\hbar\frac{|eB|}{mc}\right]\psi_\alpha(x')$$

So the eigenvalue of the operator H which are the energy values are

$$E_n = \frac{\hbar^2k^2}{2m} + \left(n + \frac{1}{2}\right)\hbar\frac{|eB|}{mc}$$

This gives the allowed energy of the charged particle. □

7.2 Homework Two

7.2.1. **(Sakurai 3.1)** Find the eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Suppose an electron is in spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If S_y is measured, what is the probability of the result $\hbar/2$?

Solution:

Suppose the eignevalues of the matrix are λ. The characterstics equation for the matrix is

$$(0-\lambda)(0-\lambda) - (-i \cdot i) = 0 \qquad \Rightarrow \lambda = \pm 1$$

Let the eigenvector be $\begin{pmatrix} x \\ y \end{pmatrix}$. Then the eigenvector corresponding to $\lambda = 1$ we have

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -iy & = x \\ ix & = y \end{matrix} \quad \Rightarrow \quad \begin{matrix} x & = 1 \\ y & = i \end{matrix}$$

Normalizing this eivenvector we have the normalization factor $\sqrt{1^2+1^2} = \sqrt{2}$. So the required normalized eigenvector corresponding to $\lambda = 1$ is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Then the eigenvector corresponding to $\lambda = -1$ we have

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -iy & = -x \\ ix & = -y \end{matrix} \quad \Rightarrow \quad \begin{matrix} x & = & 1 \\ y & = & -i \end{matrix}$$

Normalizing this eivenvector we have the normalization factor $\sqrt{1^2+1^2} = \sqrt{2}$. So the required normalized eigenvector corresponding to $\lambda = -1$ is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

So the eigenvectors corresponding to each eigenvalues are

$$\lambda = 1 \quad \rightarrow \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \lambda = -1 \quad \rightarrow \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Let the arbitrary spin state be $|\gamma\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ such that its dual correspondence is $\langle\gamma| = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$. Since the matrix representation of the S_y operator is $\frac{\hbar}{2}\sigma_y$. The probability that the state be measure to be in S_y with eigenvalue $\frac{\hbar}{2}$ is

$$\langle\gamma|\hbar/2\sigma_2|\gamma\rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \frac{i\hbar}{2}\left(-\beta\alpha^* + \beta^*\alpha\right)$$

So the probability of measuring the given state in $|S_y;+\rangle$ state is $\frac{i\hbar}{2}\left(\alpha\beta^* - \alpha^*\beta\right)$. □

7.2.2. **(Sakurai 3.2)** Find, by explicit construction using Pauli matrices, the eigenvalues for Hamiltonian

$$H = -\frac{2\mu}{\hbar}\boldsymbol{S}\cdot\boldsymbol{B}$$

for a spin $\frac{1}{2}$ particle in the presence a magnetic $\boldsymbol{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$.

Solution:

The hamiltonian operator in the given magnetic field as

$$H = -\frac{2\mu}{\hbar}\left(S_xB_x + S_yB_y + S_zB_z\right)$$

Since the spin operators S_x, S_y and S_z are the pauli matrices with a factor of $\hbar/2$ we can write the above expression as

$$H = -\frac{2\mu}{\hbar}\frac{\hbar}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} B_x + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} B_y + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} B_z\right]$$
$$= -\mu\begin{pmatrix} B_z & B_x - iB_y\\ B_x + iB_y & -B_z\end{pmatrix}$$

The characterstics equation for the this matrix is

$$((B_z - \lambda)(-B_z - \lambda) - (B_x - iB_y)(B_x + iB_y)) = 0 \quad \Rightarrow \lambda^2 - B_z^2 - (B_x^2 + B_y^2) = 0 \quad \Rightarrow \lambda = \pm|\boldsymbol{B}|$$

So the eigenvalue of the Hamiltonian which is $-\mu$ times the matrix is $-\mu \cdot \lambda = \mp\mu|\boldsymbol{B}|$. □

7.2.3. **(Sakurai 3.3)** Consider 2×2 matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a}}{a_0 - i\boldsymbol{\sigma}\cdot\boldsymbol{a}}$$

where a_0 is a real number and $\boldsymbol{a}$ is a three-dimensional vector with real components.

(a) Prove that U is unitary and unimodular.
Solution:
Given matrix U and hermitian conjugate can be written as

$$U = \frac{a_0 + i\sum_j a_j\sigma_j}{a_0 - i\sum_j a_j\sigma_j} \qquad U^\dagger = \frac{a_0 - i\sum_j a_j\sigma_j^\dagger}{a_0 - i\sum_j a_j\sigma_j^\dagger}$$

Multiplying these two to check for unitarity

$$U^\dagger U = \frac{a_0 - i\sum_j a_j\sigma_j^\dagger}{a_0 - i\sum_j a_j\sigma_j^\dagger}\cdot\frac{a_0 + i\sum_j a_j\sigma_j}{a_0 - i\sum_j a_j\sigma_j}$$
$$= \frac{a_0^2 + ia_0\sum_j\sigma_j a_j - ia_0\sum_j\sigma_j^\dagger a_j + \sum_j\sum_k\sigma_j^\dagger a_j\sigma_k a_k}{a_0^2 - ia_0\sum_j\sigma_j a_j + ia_0\sum_j\sigma_j^\dagger a_j + \sum_j\sum_k\sigma_j^\dagger a_j\sigma_k a_k}$$

Since each pauli matrices are Hermitian, for each i we have $\sigma_i^\dagger = \sigma_i$. This makes the numerator the exact same as the denominator. Thus they cancel out

$$U^\dagger U = \frac{a_0^2 + ia_0\sum_j\sigma_j a_j - ia_0\sum_j\sigma_j a_j + \sum_j\sum_k\sigma_j a_j\sigma_k a_k}{a_0^2 - ia_0\sum_j\sigma_j a_j + ia_0\sum_j\sigma_j a_j + \sum_j\sum_k\sigma_j a_j\sigma_k a_k} = 1$$

This shows that this matrix is unitary. Expanding out the matrix in terms of the pauli matrices we get

$$\det U = \frac{\begin{vmatrix} a_0 + ia_3 & ia_1 + a_2\\ ia_1 - a_2 & a_0 - ia_3\end{vmatrix}}{\begin{vmatrix} a_0 - ia_3 & -ia_1 + a_2\\ -ia_1 - a_2 & a_0 + ia_3\end{vmatrix}} = \frac{(a_0 + ia_3)(a_0 - ia_3) - (ia_1 + a_2)(ia_1 - a_2)}{(a_0 - ia_3)(a_0 + ia_3) - (-ia_1 + a_2)(-ia_1 - a_2)} = \frac{a_0^2 + a_1^2 + a_2^2 + a_3^2}{a_0^2 + a_1^2 + a_2^2 + a_3^2} = 1$$

This shows that the matrix is unimodular. □

(b) In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and the angle of rotation appropriate for U in terms of a_0, a_1, a_2 and a_3.

Solution:
The matrix can be rewritten as

$$U = \frac{1}{a_0^2 + \boldsymbol{a}^2} \begin{pmatrix} a_0 - \boldsymbol{a}^2 + 2ia_0a_3 & 2a_0a_2 + 2ia_0a_1 \\ -2a_0a_2 + 2ia_0a_1 & a_0 - \boldsymbol{a}^2 - 2ia_0a_3 \end{pmatrix}$$

Since the most general unimodular matrix of the form $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ represent a rotaion through an angle ϕ through the direction $\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$ related as

$$\mathrm{Re}(a) = \cos\left(\frac{\phi}{2}\right), \qquad \mathrm{Im}(a) = -n_z \sin\left(\frac{\phi}{2}\right) \tag{7.3}$$

$$\mathrm{Re}(b) = -n_y \sin\left(\frac{\phi}{2}\right), \qquad \mathrm{Im}(b) = -n_x \sin\left(\frac{\phi}{2}\right) \tag{7.4}$$

Making these comparision in this matrix we get

$$\cos\left(\frac{\phi}{2}\right) = \frac{a_0^2 - \boldsymbol{a}^2}{a_0^2 + \boldsymbol{a}^2} \qquad \Rightarrow \phi = 2\,\mathrm{acos}\left(\frac{a_0^2 - \boldsymbol{a}^2}{a_0^2 + \boldsymbol{a}^2}\right)$$

And similarly we get

$$n_x = -\frac{a_1}{|\boldsymbol{a}|} \qquad n_y = -\frac{a_2}{|\boldsymbol{a}|} \qquad n_z = -\frac{a_3}{|\boldsymbol{a}|}$$

This gives the rotation angle and the direction of rotation for this given unimodular matrix. □

7.2.4. **(Sakurai 3.9)** Consider a sequence of rotations represented by

$$\mathcal{D}^{(1/2)}(\alpha, \beta, \gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \\ e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}$$

Solution:
Again this final matrix can be written as a coplex form as

$$\mathcal{D}^{1/2}(\alpha, \beta, \gamma) = \begin{bmatrix} \left(\cos\left(\frac{\alpha+\gamma}{2}\right) + i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) & -\left(\cos\left(\frac{\alpha+\gamma}{2}\right) - i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) \\ \left(\cos\left(\frac{\alpha+\gamma}{2}\right) + i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) & -\left(\cos\left(\frac{\alpha+\gamma}{2}\right) - i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) \end{bmatrix}$$

Let ϕ be the angle of rotation represented by this final rotation matrix. Using again the equations (7.3) we get

$$\cos\left(\frac{\phi}{2}\right) = \cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right) \qquad \Rightarrow \qquad \phi = 2\cos^{-1}\left[\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right]$$

This gives the angular rotation value for this matrix. The direction of rotation can similarly be found by using (7.3) to calculate the directions. □

7.2.5. **(Sakurai 3.15a)** Let $\boldsymbol{J}$ be angular momentum. Using the fact that J_x, J_y, J_z and $J_\pm \equiv J_x \pm J_y$ satisfy the usual angular-momentum commutation relations, prove

$$\boldsymbol{J}^2 = J_z^2 + J_+J_- - \hbar J_z$$

.
Solution:

Multiplying out J_+ and J_- we get

$$\begin{aligned} J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2 \\ &= J_x^2 - i[J_x, J_y] + J_y^2 \\ &= J_x^2 + J_y^2 - i(i\hbar J_z) \\ &= \boldsymbol{J}^2 - J_z^2 + \hbar J_z \end{aligned}$$

Rearranging above expression gives $\boldsymbol{J}^2 = J_z^2 + J_+J_- - \hbar J_z$ which completes the proof. □

7.3 Homework Three

7.3.1. Expand the matrix

$$\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) = e^{-i(m'\alpha+m\gamma)}\,\langle j,m'|\exp\left(\frac{-iJ_y\beta}{\hbar}\right)|j,m\rangle\,.$$

Solution:
Clearly the order of matrix depends upon the value of j. The range of values for m are constrained by the value of j. So for $j = 1$, the matrix becomes

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

□

7.3.2. **(Sakurai 3.13)** An angular-momentum eigenstate $|j, m = m_{max} = j\rangle$ is rotated by an infinitesimal angle ε about y-axis. Withoug using the explicit form of the $d^{(j))}_{m'm}$ function, obtain an expression for the probability for a new rotated state to be found in the original state up to terms of order ε^2
Solution:
Let the given state be $|\alpha\rangle = |j,j\rangle$The rotation operator thorugh Y axis is

$$\mathcal{D}_y(\varepsilon) = \exp\left(\frac{-iJ_y\varepsilon}{\hbar}\right) = 1 - \frac{iJ_y\varepsilon}{\hbar} - \frac{J_y^2\varepsilon^2}{2\hbar^2} + \ldots$$

Writing $J_y = \frac{1}{2i}\left(J_+ - J_-\right)$ and expanding out the expression we get

$$\mathcal{D}_y(\varepsilon) = 1 - \frac{\varepsilon^2}{8\hbar^2}J_+J_-$$

So the rotetaed state is

$$|\alpha\rangle_R = \mathcal{D}_y(\varepsilon)\,|\alpha\rangle = 1 - \frac{\varepsilon^2}{8\hbar^2}J_+J_-\,|jj\rangle$$

The probability of finding the rotated state in the original state is given by $|\langle\alpha|\alpha\rangle_R|^2$ calculating thi

$$\begin{aligned} |\langle\alpha|\alpha\rangle_R|^2 = \left|\langle jj|1 - \frac{\varepsilon^2}{8\hbar^2}J_+J_-|jj\rangle\right|^2 &= \left|\langle jj|jj\rangle - \frac{\varepsilon^2}{8\hbar^2}\langle jj|J_+J_-|jj\rangle\right|^2 \\ &= \left|1 - \sqrt{2j}\hbar\sqrt{2j}\hbar\right|^2 = \left|1 - \frac{\varepsilon^2 j}{4}\right|^2 \approx 1 - \frac{\varepsilon^2 j}{2} \end{aligned}$$

This is the required probability in the order of ε^2. □

7.3.3. **(Sakurai 3.16)** Show that the orbital angular-momentum operator **L** commutes with both the operators $\mathbf{p}^2$ and $\mathbf{x}^2$

Solution:

The commutator of each component of L with $\mathbf{p}^2$ are

$$\begin{aligned}
\left[L_z, \mathbf{p}^2\right] &= \left[xp_y - yp_x, p_x^2 + p_y^2 + p_z^2\right] \\
&= \left[xp_y, p_x^2\right] - \left[yp_x, p_y^2\right] \\
&= \left(i\hbar \frac{\partial}{\partial p_x} p_x^2\right) p_y - \left(i\hbar \frac{\partial}{\partial p_y} p_y^2\right) p_x \\
&= 2i\hbar [p_x, p_y] \\
&= 0
\end{aligned}$$

Similarly we can show that this is true for every component of the **L** hence it is proved for $\left[\mathbf{L}, \mathbf{p}^2\right]$.

Now for the commutation of $\mathbf{x}^2$ witht he operator **L**

$$\begin{aligned}
\left[L_z, \mathbf{x}^2\right] &= \left[xp_y - yp_x, p_x^2 + p_y^2 + p_z^2\right] \\
&= \left[xp_y, p_y^2\right] - \left[yp_x, p_x^2\right] \\
&= x\left(-i\hbar \frac{\partial}{\partial y} y^2\right) - y\left(-i\hbar \frac{\partial}{\partial x} x^2\right) \\
&= -2i\hbar [x, y] \\
&= 0
\end{aligned}$$

Sincce this is true for the L_z component it is also true for every toehr comopnent so that the vector commutatior $\left[\mathbf{L}, \mathbf{x}^2\right]$

□

7.3.4. **(Sakurai eq 3.6.11)** Prove the following

(a)

$$\langle \boldsymbol{x'} | L_x | \alpha \rangle = -i\hbar \left(-\sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi}\right) \langle \boldsymbol{x'} | \alpha \rangle$$

(b)

$$\langle \boldsymbol{x'} | L_y | \alpha \rangle = -i\hbar \left(\cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}\right) \langle \boldsymbol{x'} | \alpha \rangle$$

(c)

$$\langle \boldsymbol{x'} | \mathbf{L}^2 | \alpha \rangle = -\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin \frac{\partial}{\partial \theta}\right)\right] \langle \boldsymbol{x'} | \alpha \rangle$$

Solution:

The angular momentum operator is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{P} = \mathbf{r} \times (-i\hbar\nabla) = (-i\hbar) r \times \nabla$$

These vectors in sperical coordinate system are

$$\boldsymbol{r} = r\hat{\mathbf{r}} + \theta\hat{\theta} + \phi\hat{\phi} \qquad \nabla = \hat{\mathbf{r}}\theta \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi}$$

So teh corss product is

$$\mathbf{L} = (-i\hbar)\mathbf{r} \times \nabla = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\theta} & \hat{\phi} \\ r & \theta & \phi \\ r\frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{vmatrix} = (-\hbar)\left(\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)$$

Now the cartesian unit vectors in the spherical unit vectors are

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\sin\phi$$

Thus the angular momentum operator in the L_x direction becomes

$$L_x = \hat{\mathbf{x}} \cdot \mathbf{L} = (-i\hbar)\left(-\sin\phi\frac{\partial}{\partial \theta} - \cos\phi\cot\theta\frac{\partial}{\partial \phi}\right)$$

Thus

$$\langle \boldsymbol{x}|L_x|\alpha\rangle = -(-i\hbar)\left(-\sin\phi\frac{\partial}{\partial \theta} - \cos\phi\cot\theta\frac{\partial}{\partial \phi}\right)\langle \boldsymbol{x}|\alpha\rangle$$

Similarly the operator L_y is

$$L_y = \hat{\mathbf{y}} \cdot \mathbf{L} = (-i\hbar)\left(\cos\phi\frac{\partial}{\partial \theta} - \sin\phi\cot\theta\frac{\partial}{\partial \phi}\right)$$

Thus

$$\langle \boldsymbol{x}|L_y|\alpha\rangle = -(-i\hbar)\left(-\cos\phi\frac{\partial}{\partial \theta} - \sin\phi\cot\theta\frac{\partial}{\partial \phi}\right)\langle \boldsymbol{x}|\alpha\rangle$$

Also the angular momentum squared opeator becomes

$$\begin{aligned} \mathbf{L}^2 = \mathbf{L}\cdot\mathbf{L} &= \left[(-\hbar)\left(\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)\right] \cdot \left[(-\hbar)\left(\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)\right] \\ &= -\hbar^2\left[\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right] \\ &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right] \end{aligned}$$

Thus we can write

$$\langle \boldsymbol{x}|\mathbf{L}^2|\alpha\rangle = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right]\langle \boldsymbol{x}|alpha\rangle$$

These are the required operator representation in spherical coordinate system. □

7.4 Homework Four

7.4.1. **(Sakurai 3.18)** A particle in a spherically symmetrical potential is known to be in an eigenstat of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively. Prove that the expectation values between $|lm\rangle$ states satisfy

$$\langle L_x\rangle = \langle L_y\rangle = 0, \qquad \langle L_x^2\rangle = \langle L_y^2\rangle = \frac{\left[l(l+1)\hbar^2 - m^2\hbar^2\right]}{2}$$

Intrepret the result semiclassically.
Solution:
Since the definition of the operators $L_\pm = L_x \pm L_y$ these relations can be rearranged in to the oprators the operators

$$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$$

The expectation value of operator L_x is

$$\begin{aligned} \langle L_x \rangle = \langle lm | L_x | lm \rangle &= \langle lm | \frac{L_+ + L_-}{2} | lm \rangle \\ &= \frac{1}{2} \langle lm | L_+ | lm \rangle + \frac{1}{2} \langle lm | L_- | lm \rangle \\ &= \frac{1}{2} \langle lm | C_+ | lm+1 \rangle + \frac{1}{2} \langle lm | C_- | lm+1 \rangle \\ &= 0 + 0 = 0 \end{aligned}$$

Similarly for L_y the expectation value is zero. The L_x^2 opeartor can be expanded into

$$\begin{aligned} L_x^2 &= \left[\frac{L_+ + L_-}{2} \right] \left[\frac{L_+ + L_-}{2} \right] \\ &= \frac{1}{4} \left(L_+^2 + L_+ L_- + L_- L_+ + L_-^2 \right) \end{aligned}$$

But the expectattion value of L_+^2 and L_-^2 are both zero because they raise and lower the state ket twice which are othognoal to each other.

Now the expectation value reduces to

$$\left\langle L_x^2 \right\rangle = \frac{1}{4} \langle L_+ L_- + L_- L_+ \rangle$$

But

$$L_+ L_- + L_- L_+ = L_x^2 - iL_x L_y + iL_y L_x + L_y^2 + L_x^2 + iL_x L_y - iL_y L_x + L_y^2 = 2(L_x^2 + L_x^2) = 2(\mathbf{L}^2 - L_z^2)$$

Using this to find the expectation value of L_x^2 we get

$$\left\langle L_x^2 \right\rangle = \frac{1}{4} \langle L_+ L_- + L_- L_+ \rangle = \frac{1}{2} \left\langle \mathbf{L}^2 - L_2^2 \right\rangle = \frac{1}{2} \left(\hbar^2 l(l+1) + \hbar^2 m^2 \right)$$

Similarly the expectatin value of L_y^2 is same as for L_x^2 and they are arequal. □

7.4.2. **(Sakurai 3.19)** Suppose a half-integer $l$value, say $\frac{1}{2}$, were allowed for orbital angular momentum. From

$$L_+ Y_{1/2,1/2}(\theta, \phi) = 0$$

we may deduce, as usual

$$Y_{1/2,1/2}(\theta, \phi) \propto e^{i\phi/2} \sqrt{\sin\theta}$$

Now try to construct $Y_{1/2,-1/2}(\theta, \phi)$ by (a) applying L_- to $Y_{1/2,1/2(\theta,\phi)}$; and (b) using $L_- Y_{1/2,-1/2}(\theta, \phi) = 0$. Show that the two procedures lead to contradictory result.
Solution:
Applying L_- on the given state $Y_{1/2,1/2}$ we get

$$\begin{aligned} Y_{1/2,-1/2}(\theta, \phi) &= -i\hbar e^{-i\phi} \left(i \frac{\partial}{\partial\theta} - \cot\theta \right) e^{i\phi/2} \sqrt{\sin\theta} \\ &= i\hbar e^{-i\phi} (-1) e^{-i\phi/2} \frac{1}{2} \frac{\cos\theta}{\sqrt{\sin\theta}} + i\hbar \cot\theta \frac{i}{2} e^{i\phi/2} \sqrt{\sin\theta} \\ &= -\hbar e^{-i\phi/2} \frac{\cos\theta}{\sqrt{\sin\theta}} \end{aligned}$$

checking to see if $L_{-}Y_{1/2,-1/2}(\theta,\phi)=0$

$$\begin{aligned}
L_{-}Y_{1/2,-1/2}(\theta,\phi) &= -i\hbar e^{i\phi}\left(-i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)e^{-i\phi/2}\frac{\cos\theta}{\sqrt{\sin\theta}}(-\hbar)\\
&= i\hbar^2 e^{-i\phi}\left(-i\left(-\frac{\sin\theta}{\sqrt{\sin\theta}}-\frac{1}{2}\frac{\cos^2\theta}{\sqrt{\sin^3\theta}}\right)e^{-i\phi/2}-\cot\theta\left(-i\frac{1}{2}\right)e^{-i\phi/2}\sqrt{\sin\theta}\right)\\
&= \hbar^2 e^{-3i\phi/2}\left(\frac{1}{\sqrt{\sin^3\theta}}\left[-2\sin^2\theta-\cos^2\theta+\frac{1}{2}\sin 2\theta\right]\right)
\end{aligned}$$

The last expression is not zero which contradicts our proposition that tere exists a half integer $l-$value.
□

7.4.3. **(Sakurai 3.20)** Consider an orbital angular-moemntum eigenstate $|l=2,m=0\rangle$. Suppose this sate is rotated by anangle β about $y-$axis. Find the probability for the new state to be found in $m=0,\pm1$ and ±2. (The spherical harmonics for $l=0,1$ and 2 may be useful).
Solution:
Let the arbitrary state be $|P\rangle = |l=2;m=0\rangle$ the state kaet in the rotated system is $|P\rangle_R = \mathcal{D}(0,\beta,0)\,|P\rangle$ This reotaed state can be calculated as

$$\begin{aligned}
\mathcal{D}_R(0,\beta,0)\,|P\rangle &= \sum_{m'}|l=2;m'\rangle\langle l=2;m'|\,\mathcal{D}_R(0,\beta,0)\,|l=2,m=0\rangle\\
&= \sum_{m'}|l=2;m'\rangle\,\mathcal{D}^{(2)}_{m',0}(0,\beta,0) = \sum_{m'}|l=2;m'\rangle\sqrt{\frac{4\pi}{5}}Y_2^{m'}(\beta,0)^*
\end{aligned}$$

Thus the probability of finding the rotated state same as the original stae is

$$|\langle P|\mathcal{D}_R|P\rangle|^2 = \left|\sum_{m'}\langle l=2,m=0|l=2;m'\rangle\sqrt{\frac{4\pi}{5}}Y_2^{m'}(\beta,0)^*\right|^2 = \left|\sqrt{\frac{4\pi}{5}}Y_2^{m}(\beta,0)^*\right|^2$$

This is the required probability of finding the rotated state in original state.

Now for $m=0$ we have $Y_{2,0}=\sqrt{\frac{5}{16\pi}}(3\cos^2\beta-1)$ this gives the probability $\frac{1}{4}(3\cos^2\beta-1)^2$.

For $m=\pm1$ we have $Y_{2,\pm1}=\sqrt{\frac{15}{8\pi}}(\sin\beta\cos\beta)$ this gives the probability $\frac{3}{4}\sin^2\beta\cos^2\beta$.

For $m=\pm2$ we have $Y_{2,\pm2}=\sqrt{\frac{15}{32\pi}}(\sin^2\beta)$ this gives the probability $\frac{3}{8}\sin^4\beta$. □

Chapter 8

Statistical Mechanics II

8.1 Homework One

8.1.1. Evaluate the density matrix ϱ of an electron in a magnetic field in the representation that makes $\hat{\sigma}$ diagonal. Next, show that the value of $\langle\sigma\rangle$, resulting from this representation, is precisely the same as the one obtained in class.

Solution:

The pauli spin operator σ_x is diagonal in the representation where the basis states are eivenstates of S_x operator. In S_z representation the S_x states are given by

$$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$$

The transformation operator that takes from S_z representation to S_x representation is given by operator

$$U = |S_x;+\rangle\langle+| + |S_x;-\rangle\langle-|$$

So the matrix representation of this operator is

$$U = \begin{bmatrix}\langle S_x;+|+\rangle & \langle S_x;+|-\rangle \\ \langle S_x;-|+\rangle & \langle S_x;-|-\rangle\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

The operator in the new basis can be obtained from the old basis with the transformation.

$$\sigma_z' = U^\dagger \sigma_z U = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$$

The Hamiltonian of the system in new basis is

$$\mathcal{H}' = \mu \boldsymbol{B}\cdot\boldsymbol{\sigma}' = -\mu B_z \sigma_z'$$

The density operator in cannonical ensemble is given by

$$\hat{\varrho}' = \frac{e^{-\beta\mathcal{H}}}{\mathrm{Tr}\left(e^{-\beta\mathcal{H}}\right)} \tag{8.1}$$

Carrying out the taylor expansion of the numerator in the density operator

$$\begin{aligned} e^{-\beta\mathcal{H}'} = e^{\beta\mu B_z\sigma_z'} &= 1 + \frac{\beta\mu B_z\sigma_z'}{1!} + \frac{(\beta\mu B_z\sigma_z')^2}{2!} + \frac{(\beta\mu B_z\sigma_z')^3}{3!} + \frac{(\beta\mu B_z\sigma_z')^4}{4!} + \dots \\ &= \left[1 + \frac{(\beta\mu B_z)^2}{2!} + \frac{(\beta\mu B_z)^4}{4!} + \dots\right] + \sigma_z'\left[\frac{\beta\mu B_z}{1!} + \frac{(\beta\mu B_z)^3}{3!} + \dots\right] \\ &= \cosh(\beta\mu B_z) + \sigma_z\sinh(\beta\mu B_z) \end{aligned} \tag{8.2}$$

where we have used the fact that $\sigma_z'^{2n} = 1; \quad \sigma_z'^{2n+1} = \sigma_z'$ for all n in $\{0, 1, \ldots\}$ Also we have

$$\text{Tr}(1) = 2 \qquad \text{Tr}(\sigma_z') = 0$$

So taking trace of Eq. (**??**) we get

$$\text{Tr}(e^{-\beta\mathcal{H}'}) = \text{Tr}(\cosh(\beta\mu B_z) + \sigma_z \sinh(\beta\mu B_z)) = \cosh(\beta\mu B_z)\text{Tr}(1) + \sinh(\beta\mu B_z)\text{Tr}(\sigma_z') = 2\cosh(\beta\mu B_z)$$

So the density operator (8.1) becomes

$$\hat{\varrho}' = \frac{\cosh(\beta\mu B_z) + \sigma_z \sinh(\beta\mu B_z)}{2\cosh(\beta\mu B_z)} = \frac{1}{2} + \frac{1}{2}\sigma_z' \tanh(\beta\mu B_z)$$

Now the expectation value of operator σ_z for the

$$\langle\sigma_z'\rangle = \text{Tr}(\hat{\varrho}\sigma_z') = \text{Tr}\left(\frac{1}{2}\sigma_z + \frac{1}{2}\sigma_z^2 \tanh(\beta\mu B_z)\right) = \text{Tr}\left(\frac{1}{2}\sigma_z + \frac{1}{2}\tanh(\beta\mu B_z)\right) = \tanh(\beta\mu B_z)$$

This gives the expectation value of the operator. This expression is the same as the one we obtained using the basis states where σ_z was diagonal instead of σ_x that we have here. □

8.1.2. Derive the uncertainties, $\Delta x, \Delta p$ and ΔE, of a free particle in 3D box using the density matrix expression in the coordinate representation. Then calculate the uncertainty product $\Delta x \cdot \Delta p$.

Solution:

For a particle in a box the the density matrix is given by

$$\langle r|\hat{\varrho}|r'\rangle = \frac{1}{V}\exp\left[-\frac{m}{2\beta\hbar^2}|\boldsymbol{r} - \boldsymbol{r'}|^2\right]$$

The average position of the particle is given by

$$\langle\boldsymbol{r}\rangle = \text{Tr}(\boldsymbol{r}\hat{\varrho}) = \frac{1}{V}\int \left|\exp\left[-\frac{m}{2\beta\hbar^2}|\boldsymbol{r} - \boldsymbol{r'}|^2\right] r\right|_{\boldsymbol{r}=\boldsymbol{r'}} d^3r = \frac{3}{4}R$$

The average squared position is given by

$$\langle\boldsymbol{r}^2\rangle = \text{Tr}(\boldsymbol{r}^2\hat{\varrho}) = \frac{1}{V}\int \left|\exp\left[-\frac{m}{2\beta\hbar^2}|\boldsymbol{r} - \boldsymbol{r'}|^2\right] r^2\right|_{\boldsymbol{r}=\boldsymbol{r'}} d^3r = \frac{3}{5}R^2$$

So the uncertainity in the position of particle is given by

$$\Delta r = \sqrt{\langle\boldsymbol{r}^2\rangle - \langle\boldsymbol{r}\rangle^2} = \frac{1}{4}\sqrt{\frac{3}{5}}R$$

Now the average value of momentum is given by

$$\langle\boldsymbol{p}\rangle = \text{Tr}(\boldsymbol{p}\hat{\varrho}) = \frac{-i\hbar}{V}\int \left|\frac{\partial}{\partial r}\exp\left[-\frac{m}{2\beta\hbar^2}|\boldsymbol{r} - \boldsymbol{r'}|^2\right]\right|_{\boldsymbol{r}=\boldsymbol{r'}} d^3r = -i\frac{\hbar}{V}\int 0 d^3r = 0$$

The average momentum squared is

$$\langle\boldsymbol{p}^2\rangle = \text{Tr}(\boldsymbol{p}^2\hat{\varrho}) = -\frac{\hbar^2}{V}\int \left|\frac{\partial^2}{\partial r^2}\exp\left[-\frac{m}{2\beta\hbar^2}|\boldsymbol{r} - \boldsymbol{r'}|^2\right]\right|_{\boldsymbol{r}=\boldsymbol{r'}} d^3r = 3mkT$$

Again the uncertainity in momentum is given by

$$\Delta p = \sqrt{\langle \boldsymbol{p}^2\rangle - \langle \boldsymbol{p}\rangle^2} = \sqrt{3mkT}$$

So the uncertainity product is

$$\Delta r \cdot \Delta p = \frac{3}{4}\sqrt{\frac{mkT}{5}}R$$

This gives the uncertainity product in position and momentum. □

8.1.3. Prove that

$$\langle q|e^{-\beta\mathcal{H}}|q'\rangle = \exp\left[-\beta\mathcal{H}\left(-i\hbar\frac{\partial}{\partial q}, q\right)\right]\delta(q-q'),$$

where

$$\mathcal{H}\left(-i\hbar\frac{\partial}{\partial q}, q\right)$$

is the Hamiltonian of the system in the q-representation, which formally operates upon the Dirac delta function, $\delta(q-q')$. Write δ-function is a suitable form; apply this result to a free particle.
Solution:
let $\psi_n(q) = \langle n|q\rangle$ be energy eigenfunction with eigenvalue E_n in configuration space q. Then by schrodingers equation we have

$$\mathcal{H}(-i\hbar\frac{\partial}{\partial q}, q)\psi_n(q') = E_n\psi_n(q')$$

Since we know that for operators $A\psi(x) = \lambda\phi(x) \implies f(A)\phi(x) = f(\lambda)\phi(x)$

$$e^{-\beta\mathcal{H}(-i\hbar\frac{\partial}{\partial q}, q)}\psi_n(q') = e^{-\beta E_n}\psi_n(q')$$

This can be used to write

$$\begin{aligned}\langle q|e^{-\beta\mathcal{H}}|q'\rangle &= \sum_n \langle q|n\rangle\ \langle n|e^{-\beta\mathcal{H}}|q'\rangle && \left(\text{Inserting } \sum_n |n\rangle\langle n|\right)\\ &= \sum_n \psi_n(q)e^{-\beta E_n}\psi_n^*(q')\\ &= e^{\mathcal{H}(-i\hbar\frac{\partial}{\partial q}, q)}\sum_n \psi_n(q)\psi_n^*(q')\end{aligned}$$

But since the the eigenfunctions of the Hamiltonian are orthogonal to each other we get $\sum_n \psi^*(q')\psi(q) = \delta(q-q')$ we get

$$\langle q|e^{-\beta\mathcal{H}}|q'\rangle = e^{\mathcal{H}(-i\hbar\frac{\partial}{\partial q}, q)}\delta(q-q') \tag{8.3}$$

This is the required expression for the matrix element of the dnesity operator $e^{-\beta\mathcal{H}}$.

We can also write the δ–function using the fourier transform representaation of δ–function as

$$\delta(q-q') = \left(\frac{1}{2\pi}\right)^3\int_{-\infty}^{\infty} e^{ik(q-q')}dk \tag{8.4}$$

For a free particle the Hamiltonian can be written as

$$\mathcal{H}(-i\hbar\frac{\partial}{\partial q}, q) = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q\prime^2} \tag{8.5}$$

Now using (8.5) and (8.4) this in (8.3) we get

$$\begin{aligned}
\langle q|e^{-\beta\mathcal{H}}|q'\rangle &= \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} e^{\mathcal{H}(-i\hbar\frac{\partial}{\partial q}, q)} e^{ik(q-q')} dk \\
&= \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} e^{-\frac{\beta\hbar^2}{2m}+ik(q-q')} dk
\end{aligned}$$

This can be solved by completing the square in the exponential and using the gamma function the final result is

$$\langle q|e^{-\beta\mathcal{H}}|q'\rangle = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{3}{2}} e^{-\frac{m}{2\beta\hbar^2}(q-q')^2}$$

This is the matrix element of the density operator for the free particle in a box. □

8.1.4. Derive the density matrix ρ for a free particle in the momentum representation and study its main properties, such as the average energy, momentum.

Solution:

The Hamiltonian of the free particle in moementum representation is

$$\mathcal{H} = \frac{\hat{p}^2}{2m}$$

Let $|\psi_k\rangle$ be the momentum wavefunction of the particle then the expression for the momentum wavefunction is

$$\psi_k(r) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

Since the momentum eigenfunctions make complete set of states they are orthonormal

$$\langle\psi_k|\psi'_k\rangle = \delta_{k,k'}$$

Now the cannonical partition function of the system is

$$\begin{aligned}
Q(V,T) &= \mathrm{Tr} e^{-\beta\mathcal{H}} \\
&= \sum_k \langle\psi_k|e^{-\beta\mathcal{H}}|\psi_k\rangle \\
&= \sum_k e^{-\frac{\beta\hbar^2}{2m}k^2}
\end{aligned}$$

Since the states are very close in momentum sapce we can replace the sum by integral

$$\begin{aligned}
Q(V,T) &= \frac{V}{(2\pi)^3}\int dK e^{-\frac{\beta\hbar^2}{2m}k^2} \\
&= \frac{V}{(2\pi)^3}\left(\frac{2m\pi}{\beta\hbar^2}\right)^{\frac{3}{2}} \\
&= \frac{V}{\lambda^3}
\end{aligned}$$

The matrix element of this operator now become

$$\langle \psi_k | \hat{\varrho} | \psi_k' \rangle = \frac{\lambda^3}{V} e^{-\frac{\beta\hbar^2}{2m}k^2} \delta_{k,k'}$$

Ths is the requried density matrix representation in momentum sapce. □

8.1.5. We showed in class that linearly polarized light corresponds to apure state and non-polarized light is in a mixed state. What is the circularly polarized, a mixed state or a pure state? Verify your statement
Solution:
Polarized light must be pure state because, at any given time it only has components The two plane polarized components x be represented by A $\begin{bmatrix}1\\0\end{bmatrix}$ a and y plane polarized be represented by $\begin{bmatrix}0\\1\end{bmatrix}$. The most general polarization of the light can be written as the linear combination of these two plane polarized conponents as

$$P_{\text{gen}} = a \begin{bmatrix}1\\0\end{bmatrix} e^{i\theta_1} + \begin{bmatrix}0\\1\end{bmatrix} e^{i\theta_2}$$

where a and b in general are complex numbers. For a circularly polarize. If the two plane polarized components have a total phase difference of $n\pi$ then the light is plane polarized. But for the phase difference $\delta = \theta_2 - \theta_1 = (2n+1)\frac{\pi}{2}$ the light is circularly polarized. Let the phase $\theta_1 = 0$ and $\theta_2 = \pi/2$ such the phase difference is $\pi/2$ we get

$$P_{\text{circular}} = \frac{1}{\sqrt{2}} \begin{bmatrix}1\\0\end{bmatrix} + \frac{i}{\sqrt{2}} \begin{bmatrix}0\\1\end{bmatrix}$$

Now for this representation, the density matrix can be obtained easily as

$$\hat{\varrho} = \begin{bmatrix} aa^* & ab^* \\ ba^* & bb^* \end{bmatrix} = \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix}$$

This prepresents a pure state as

$$\hat{\varrho}^2 = \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} \times \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} = \hat{\varrho}$$

This verifies that the circularly polarized light is pure state. □

8.2 Homework Two

8.2.1. We mentioned in class that in calculating the matrix of $e^{-\beta\mathcal{H}}$, $\langle 1,2,3,N | e^{-\beta\mathcal{H}} | 1,2,3,N \rangle$, permutation goth the particle coordinates in the first wave function and energy states in the second yields a result which is N! of the result for a fixed set of $\{k,\}$ states that is, without permuting the energy states. Do it explicitly of two particle and two state case starting with $u_a(1)u_b(2)$.
Solution:
The general matrix element for N particle n state system from Pathria eq (5.5.12) is

$$\langle 1,\ldots,N | e^{-\beta\mathcal{H}} | 1',\ldots,N' \rangle = \frac{1}{N!} \sum_k e^{-\frac{\beta\hbar^2 k^2}{2m}} \left[\sum_p \delta_p \left\{ u_{k_1}(p_1) \ldots u_{kn} \right\} \right] \ldots \left[\sum_p \delta_p \left\{ u^*_{k_1}(p_1) \ldots u^*_{kn} \right\} \right]$$

For two particle and two sate we get

$$\langle 1,2 | e^{-\beta\mathcal{H}} | 1',2' \rangle = \frac{1}{2!} \sum_k e^{-\frac{\beta\hbar^2 k^2}{2m}} \left[u_a(1)u_b(2) \pm u_a(2)u_b(1) \right] \left[u^*_a(1)u^*_b(2) \pm u^*_a(2)u^*_b(1) \right]$$

Multiplying the wavefunctions we get

$$\langle 1,2|e^{-\beta\mathcal{H}}|1\prime,2\prime\rangle = \frac{1}{2!}\sum_k e^{-\frac{\beta\hbar^2k^2}{2m}}\left[u_a(1)u_b(2)u_a^*(1)u_b^*(2) \pm u_a(1)u_b(2)u_a^*(1)u_b^*(2)\right.$$
$$\left. + u_a(1)u_b(2)u_a^*(1)u_b^*(2) \pm u_a(1)u_b(2)u_a^*(1)u_b^*(2)\right]$$

For the case of fixed $\{k_i\}$, i.e., if only the particles are permuted

$$\langle 1,2|e^{-\beta\mathcal{H}}|1',2'\rangle = \frac{1}{2!}\sum_k e^{-\frac{\beta\hbar^2k^2}{2m}}\left[u_a(1)u_b(2)u_a^*(1)u_b^*(2) \pm u_a(1)u_b(2)u_a^*(1)u_b^*(2)\right]$$

But since the density operator is hermition, the matrix elements are equal to the complex conjugate of itself with the coordinate exchanged

$$\langle 1,2|e^{-\beta\mathcal{H}}|1',2'\rangle = \langle 1,2|e^{-\beta\mathcal{H}}|2',1'\rangle^*$$

This would essentially mean

$$u_a(1)u_a^*(1')u_b(2)u_b^*(2') = u_a(2)u_a^*(2')u_b(1)u_b^*(1')$$
$$u_a(2)u_a^*(1')u_b(1)u_b^*(2') = u_a(1)u_a^*(2')u_b(1)u_b^*(2')$$

Using this in the sum we get

$$\langle 1,2|e^{-\beta\mathcal{H}}|1',2'\rangle = \frac{1}{2!}\sum_k e^{-\frac{\beta\hbar^2k^2}{2m}}\left[u_a(1)u_b(2) \pm u_a(2)u_b(1)\right]\left[u_a^*(1)u_b^*(2) \pm u_a^*(2)u_b^*(1)\right]$$
$$= \sum_k e^{-\frac{\beta\hbar^2k^2}{2m}}\left[u_a(1)u_b(2)u_a^*(1)u_b^*(2) \pm u_a(1)u_b(2)u_a^*(1)u_b^*(2)\right]$$

Here the last expression is exactly twice the expression for fixed $\{k_i\}$case. Where 2 is equal to the factorial of itself $2! = 2$ thus the rusult is $N!$ times the expression for fixed $\{k_i\}$ case. □

8.2.2. Study the density matrix and the partition function of a system of free particles, using un symmetrized wave function instead of symmetried wave function. Show that, following the text procedure, on encounters neighter th Gibbs' correction factor $\frac{1}{N!}$ nor a spatial correlation among the particles.
Solution:
If we used unsymmetrized wave function rather than symmetrized wave function we get

$$\langle 1,2,\ldots,N|e^{-\beta\mathcal{H}}|1,2,\ldots,N\rangle = \sum_k e^{-\beta\frac{\hbar^2k^2}{2m}}(u_{k_1}(1)\ldots u_{k_n}(N))(u_{k_1}^*(1')\ldots u_{k_N}^*(N'))$$
$$= \sum_{k_1,\ldots k_N} e^{\beta\hbar^2\frac{k_1^2+\ldots+k_N^2}{2m}}(u_{k_1}(1)\ldots u_{k_n}(N))(u_{k_1}^*(1')\ldots u_{k_N}^*(N'))$$

The summation in the exponential can now be changed into product of the exponential and the expression becomes

$$= \prod_{i=1}^{N}\left[e^{-\beta\hbar^2/2m}\left\{u_{k_i}(i)u_{k_j}^*(j')\right\}\right]$$

Since the states are dense we can change the summation over k_i by the integration

$$\langle 1,2,\ldots,N|e^{-\beta\mathcal{H}}|1,2,\ldots,N\rangle = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{3N}{2}}\exp\left(-\frac{m}{2\beta\hbar^2}\left(|r_1-r_2'|^2+\ldots|r_N-r_N'|^2\right)\right)$$

From this expression its easy to calculate the diagonal elements, because for diagonal elements we have $r_i = r'_i$. This makes the exponential identically equal to zero and we get the matrix element

$$\langle 1, 2, \ldots, N | e^{-\beta \mathcal{H}} | 1, 2, \ldots, N \rangle = \left(\frac{m}{2\pi\beta\hbar^2} \right)^{\frac{3N}{2}}$$

Using the wavelength parameter

$$\lambda = \sqrt{\frac{m}{2\pi\beta\hbar^2}}$$

we get the Matrix element as

$$\langle 1, 2, \ldots, N | e^{-\beta \mathcal{H}} | 1, 2, \ldots, N \rangle = \left(\frac{1}{\lambda} \right)^{3N}$$

Now the cannonical partition function is just the trace of this expression

$$Q_N(T, V) = \mathrm{Tr}(e^{-\beta \mathcal{H}}) = \int \left(\frac{1}{\lambda} \right)^{3N} d^{3N} r = \left(\frac{V}{\lambda^3} \right)^N$$

This expression has neigher the gibbs correction factor $\frac{1}{N!}$ nor the spatial correction factor. □

8.2.3. Determine the values of the degeneracy discriminant $n\lambda^3$ for hydrogen, helium and oxygen at NTP. Make an estimate of the respectivey temperature ranges where the magnitues of this quantity becomes coparamble to unity and hence quantum effects become important.

Solution:

The quantity $n\lambda^3$ can be written in terms of temperature and boltzman constant as

$$n\lambda^3 = \frac{nh^3}{(2\pi mkT)^{3/2}} = \frac{N}{V} \frac{h^3}{(2\pi mkT)^{3/2}} = \frac{h^3 P}{(2\pi m)^{3/2} (kT)^{5/2}} \tag{8.6}$$

For standard temperature and pressure

$$T = 293 K and P = 1.01 \times 10^5$$

Using the mass of Hiydrogen, Helium and Oxygen we get

$$H_2 : n\lambda^3 = \frac{6.63 \times 10^{-34} 1.01 \times 10^5}{2\pi(1.67 \times 10^{-27})^{3/2} (1.38 \times 10^{-23} \times 293)^{5/2}} = 2.86 \times 10^{-5}$$

$$He_2 : n\lambda^3 = \frac{6.63 \times 10^{-34} 1.01 \times 10^5}{2\pi(6.64 \times 10^{-27})^{3/2} (1.38 \times 10^{-23} \times 293)^{5/2}} = 3.61 \times 10^{-6}$$

$$O_2 : n\lambda^3 = \frac{6.63 \times 10^{-34} 1.01 \times 10^5}{2\pi(25.6 \times 10^{-27})^{3/2} (1.38 \times 10^{-23} \times 293)^{5/2}} = 4.78 \times 10^{-7}$$

INverting the relation (8.6) and setting $n\lambda^3 \simeq 1$ we get

$$T = \frac{1}{K} \left(\frac{h^6 P^2}{(2\pi m)^3} \right)^{1/5}$$

So for the different masses of H2, He2 and O2 we get

$$H_2 : T = 4.46 K$$
$$He_2 : T = 1.95 K$$
$$O_2 : T = 0.868 K$$

This give the temperature in which the discriminant is close to 1. □

8.2.4. A system consists of three particles, each of which has three possible quantum states, which energy 0 , $2E$, or $5E$ respectively. Write out the complete expression of the cannonical partition function Q for this system:

(a) if the articles obey Maxwells-Boltzman statistics.
Solution:
The single particle cannonical parition function for

$$Q_1(V,T) = \sum_n e^{-\beta E_n} = 1 + e^{-2\beta} + e^{-5\beta}$$

The cannonical partition function for N distinguishable particles is obtained by $Q_N(V,T) = \frac{1}{N!}\left[Q_1(V,T)\right]^N$ So for three particles we get

$$Q_3(V,T) = \frac{1}{3!}\left[1 + e^{-2\beta} + e^{-5\beta}\right]^3$$

The free energy of the system is

$$F = kT\ln Q = kT\ln\left(\frac{1}{6}\left[1 + e^{-2\beta E} + e^{-5\beta E}\right]^3\right) = -kT\ln 6 + 3kT\ln\left(1 + e^{-2\beta E} + e^{-5\beta E}\right)$$

The entropy is given by

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T}\right)_{N,V} \\ &= \frac{Tk\left(\frac{6Ee^{-\frac{2E}{Tk}}}{T^2k} + \frac{15Ee^{-\frac{5E}{Tk}}}{T^2k}\right)}{1 + e^{-\frac{2E}{Tk}} + e^{-\frac{5E}{Tk}}} + k\ln\left(\frac{\left(1 + e^{-\frac{2E}{Tk}} + e^{-\frac{5E}{Tk}}\right)^3}{6}\right) \end{aligned}$$

This gives the expression for the entropy of the particles. □

(b) if they obey Bose-Einstein statistics,
Solution:
For bose einstein case, the particle sare counted indistinguishable. So each of the three particle can belong to following energy state So the total partition function of the system becomes

n0,n1,n2	5,0,2	5,5,2	5,5,0	5,2,2	0,2,2	5,0,0	0,2,0	5,5,5	2,2,2	0,0,0
Total Energy	7E	12E	10E	9E	4E	5E	2E	15 E	6E	0

$$Q_N(T,V) = 1 + e^{-2E\beta} + e^{-4E\beta} + e^{-5E\beta} + e^{-6E\beta} + e^{-7E\beta} + e^{-9E\beta} + e^{-10E\beta} + e^{-12E\beta} + e^{-15E\beta}$$

Similarly the free energy is given by $F = kT\ln Q_N(V,T)$ and the entropy is given by $S = -\frac{\partial F}{\partial T}$
This gives the expression for the entropy of the particles. □

(c) if they obey Fermi-Dirac statistics,
Solution:
For the particle satisfying Fermi-Dirac statistics no two particles can occupy the same energy levels so each has to sit on its own energy leven which gives the partition function

$$Q_N(V,T) = \left[1 + e^{-2\beta E} + e^{-5\beta E}\right]$$

The free energy of the system is

$$F = Tk\log\left(1 + e^{-\frac{2E}{Tk}} + e^{-\frac{5E}{Tk}}\right)$$

So the entropy becomes

$$S = -\frac{\partial F}{\partial T} = -\frac{E\left(2e^{3E\beta}+5\right)}{T\left(e^{5E\beta}+e^{3E\beta}+1\right)} - k\log\left(1+e^{-2E\beta}+e^{-5E\beta}\right)$$

This gives the entropy of particles for Fermi-Dirac statistics. □

8.3 Homework Three

8.3.1. **(Pathria and Beale 6.1)** Show that the entropy of an ideal gas in thermal equilibrium is given by the formula

$$S = k\sum_{\varepsilon}\left[\langle n_\varepsilon + 1\rangle \ln\langle n_\varepsilon + 1\rangle - \langle n_\varepsilon\rangle \ln\langle n_\varepsilon\rangle\right]$$

in the case of bosons and by the formula

$$S = k\sum_{\varepsilon}\left[-\langle 1-n_\varepsilon\rangle \ln\langle 1-n_\varepsilon\rangle - \langle n_\varepsilon\rangle \ln\langle n_\varepsilon\rangle\right]$$

in the case of fermions. Verify that these results are consistent with the general formula

$$S = -k\sum_{\varepsilon}\left\{\sum_{n} p_\varepsilon(n)\ln p_\varepsilon(n)\right\},$$

where $p_\varepsilon(n)$ is the probability that there are exactly n particles in the energy stare ε.

Solution:

The general form of entropy of of the system is given by

$$S = K\sum_i\left[n_i^*\ln\left(\frac{g_i}{n_i^*}\right) + \left(n_i^* - \frac{g_i}{a}\right)\ln\left(1 - a\frac{n_i s^*}{g_i}\right)\right]$$

where, n_i^* is the set confirming to most probable distribution among the cells. With the degeneracy factor $g_i = 1$, we get $\frac{n_i}{g_i} = n_i^*$ Also the average n_ε is given by

$$\langle n_\varepsilon\rangle = z\left(\frac{\partial q}{\partial z}\right)_{V,T} = \frac{1}{z^{-1}e^{-\beta\epsilon}+a} = n_i^*$$

Substituting $n_i^* = \langle n_\varepsilon\rangle$ we get

$$S = k\sum_{\varepsilon}\left[-\langle n_\varepsilon\rangle\ln\langle n_\varepsilon\rangle + \left(\langle n_\varepsilon\rangle - \frac{1}{a}\right) ln\left(1 - a\langle n_\varepsilon\rangle\right)\right]$$

Now for bosons $a = -1$, we get

$$S = k\sum_{\varepsilon}\left[-\langle n_\varepsilon\rangle\ln\langle n_\varepsilon\rangle + \left(\langle n_\varepsilon\rangle + 1\right) ln\left(1 + \langle n_\varepsilon\rangle\right)\right]$$

Which is the required expression for the bosons. Now for fermions we substitute $a = 1$ and obtain To show that the general expression

$$S = -k \sum_{\varepsilon} \left\{ \sum_{n} p_{\varepsilon}(n) \ln p_{\varepsilon}(n) \right\},$$

works for the entropy we first notice that the expression can be modified rewritten as

$$S = -k \sum_{\varepsilon} \langle \ln p_{\varepsilon}(n) \rangle$$

Also for bosons the probability of having exactly n particle in the state with energy ε is given by

$$p_{\varepsilon}(n) = \frac{\langle n_{\varepsilon} \rangle^{n}}{(\langle n_{\varepsilon} \rangle + 1)^{n+1}} \tag{8.7}$$

$$\ln p_{\varepsilon}(n) = n \ln \langle n_{\varepsilon} \rangle - (1+n) ln \left(\langle n_{\varepsilon} \rangle + 1 \right) \tag{8.8}$$

Now substituting this to the general expression of entropy the inner summ over all n becomes

$$\begin{aligned} S &= -k \sum_{\varepsilon} \langle n \ln \langle n_{\varepsilon} \rangle - (1+n) ln \left(\langle n_{\varepsilon} \rangle + 1 \right) \rangle \\ &= -k \sum_{\varepsilon} \langle n_{\varepsilon} \rangle \ln \langle n_{\varepsilon} \rangle - (1 + \langle n_{\varepsilon} \rangle) ln \left(\langle n_{\varepsilon} \rangle + 1 \right) \\ &= k \sum_{\varepsilon} \left[- \langle n_{\varepsilon} \rangle \ln \langle n_{\varepsilon} \rangle + \left(\langle n_{\varepsilon} \rangle + 1 \right) ln \left(1 + \langle n_{\varepsilon} \rangle \right) \right] \end{aligned}$$

which shows that the general expression is true for bosons.

Substituting $a = 1$ for fermions we get

$$S = k \sum_{\varepsilon} \left[- \langle n_{\varepsilon} \rangle \ln \langle n_{\varepsilon} \rangle + \left(\langle n_{\varepsilon} \rangle - 1 \right) ln \left(1 - \langle n_{\varepsilon} \rangle \right) \right]$$

Which is the required expression for the fermions entropy.

The probability of faving exactly $n = \{0, 1\}$ particles in the cell for fermoins is given by

$$p_{\varepsilon}(n) = \begin{cases} 1 - \langle n_{\varepsilon} \rangle & \text{if } n = 0 \\ \langle n_{\varepsilon} \rangle & \text{if } n = 1 \end{cases}$$

This gives only two terms in the inner sum of the general expression so

$$S = -k \sum_{\varepsilon} \left[\langle n_{\varepsilon} \rangle ln \langle n_{\varepsilon} \rangle + (1 - \langle n_{\varepsilon} \rangle \ln(1 - \langle n_{\varepsilon} \rangle)) \right]$$

Which shows that the general expression holds for fermions too. □

8.3.2. **(Pathria and Beale 6.2)** Derive for all three statistics, the relevant expressions for the quantity

$$\langle n_{\varepsilon}^{2} \rangle - \langle n_{\varepsilon} \rangle^{2} = kT \left(\frac{\partial \langle n_{\varepsilon} \rangle}{\partial \mu} \right)_{T}$$

Compare with the previous results that we showed in class,

$$\langle n^{2} \rangle - \langle n \rangle^{2} = kT \left(\frac{\partial \langle n \rangle}{\partial \mu} \right)_{T}$$

for a system embedded in a grand canonical ensemble.

Solution:

This problem is the find the first and second moments of n_ε and their difference. Once we know the probability mass function (pmf) of the variable finding moment quite generally is

$$\langle f(x)\rangle = \sum_x f(x)p(x)$$

where $p(x)$ is the pmf. Now for the bosons, (8.9) can be slightly rewritten as

$$p_\varepsilon(n) = \frac{\langle n_\varepsilon\rangle^n}{(\langle n_\varepsilon\rangle+1)^{n+1}} = \frac{1}{\langle n_\varepsilon\rangle+1}\frac{\langle n_\varepsilon\rangle^n}{(\langle n_\varepsilon\rangle+1)^n} = \left(1-\frac{\langle n_\varepsilon\rangle}{1+\langle n_\varepsilon\rangle}\right)\left(\frac{\langle n_\varepsilon\rangle}{\langle n_\varepsilon\rangle+1}\right)^n \tag{8.9}$$

With substitution $\frac{\langle n_\varepsilon\rangle}{1+\langle n_\varepsilon\rangle} = t$ we get

$$p(n) = (1-t)t^n$$

Now the first moment of this pmf is

$$\langle n\rangle = \sum_{n=0}^{\infty} n(1-t)t^n = (1-t)\frac{t}{(1-t)^2} = \frac{t}{1-t} \qquad\qquad \because \frac{1}{(1-t)^2} = \sum_{n=0}^{\infty} nt^{n-1}$$

Similarly the second moment is

$$\langle n^2\rangle = \sum_{n=0}^{\infty} n^2(1-t)t^n = (1-t)\frac{t(1+t)}{(1-t)^3} = \frac{t(1+t)}{(1-t)^2} \qquad\qquad \because$$

Thus the variance is

$$\langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2 = \frac{t(1+t)}{(1-t)^2} - \frac{t^2}{(1-t)^2} = \frac{t}{(1-t)^2}$$

Now substuting back the value of t we get

$$\langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2 = \langle n_\varepsilon\rangle + \langle n_\varepsilon\rangle^2$$

For the Fermions we get

$$\langle n_\varepsilon^2\rangle = \sum_{n=0}^{1} n^2 p_\varepsilon(n) = p_\varepsilon(1) = \langle n_\varepsilon\rangle$$

This implies tht the variance is

$$\langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2 = \langle n_\varepsilon\rangle - \langle n_\varepsilon\rangle^2$$

For Boltzmann particle the pmf is a poisson distribution

$$p_\varepsilon(n) = \frac{\langle n_\varepsilon\rangle^n e^{-\langle n_\varepsilon\rangle}}{n!}$$

For poisson distribution is it can be easily shown that the mean and variance is just the parameter $\langle n_\varepsilon\rangle$ Thus we have

$$\langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2 = \langle n_\varepsilon\rangle$$

Looking at each of these three variances we see that it is of the general from

$$\langle n_\varepsilon^2 \rangle - \langle n_\varepsilon \rangle^2 = \langle n_\varepsilon \rangle - a \langle n_\varepsilon \rangle^2$$

Also the expectation value $\langle n_\varepsilon \rangle$ is given by

$$\langle n_\varepsilon \rangle = \frac{1}{z^{-1} e^{\beta\varepsilon} + a}$$

Differentiating this with respect to μ at constant temperature we get

$$\left[\frac{\partial \langle n_\varepsilon \rangle}{\partial \mu}\right]_T = \frac{\langle n_\varepsilon \rangle^2}{kT} \left[\frac{1}{\langle n_\varepsilon \rangle} - a\right]$$

Rearranging we get

$$KT \left[\frac{\partial \langle n_\varepsilon \rangle}{\partial \mu}\right]_T = - \langle n_\varepsilon \rangle - a \langle n_\varepsilon \rangle^2$$

Now the comparision of this expression for all the statistics leads to

$$\langle n_\varepsilon^2 \rangle - \langle n_\varepsilon \rangle^2 = KT \left[\frac{\partial \langle n_\varepsilon \rangle}{\partial \mu}\right]_T$$

This expression is true in general for all statistics. □

8.3.3. **(K. Huang 8.6)** What is the equilibrium ratio of ortho- to para-hydrogen at a temperature of 300 K? What is this ratio in the limit of high temperature? Assume that the distance between the protons in the molecule is 0.74 Angstrom.
Solution:
The equilibrium ratio is given by

$$\frac{N_{\text{ortho}}}{N_{\text{para}}} = 3 \frac{\sum_{n=odd} (2n+1) e^{-\beta\hbar^2/2Il(l+1)}}{\sum_{n=even} (2n+1) e^{-\beta\hbar^2/2Il(l+1)}}$$

Evaluating this sum explicitly with series method we get For large values of n the ratio go to one because for large n the two quantities in Numerator and denomenator are essentially the same. So we get

$$\frac{N_{\text{ortho}}}{N_{\text{para}}} = 3$$

This gives the equilibrium ratio of ortho and para hydrogen in the temperature required. □

8.3.4. Consider the thermal properties of conducting electrons in a metal and treat electrons as non-interacting particles, when particle density is high. Assuming each Cu atom donates an electron to the conducting electron gas, calculate the chemical potential, or the Fermi energy, of copper, for which the mass density is $\frac{9g}{cm^3}$. Express your answer in Kelvin.
Solution:
The fermi energy is given by

$$E_f = \frac{h^2}{8m} \left(\frac{3N}{\pi V}\right)^{\frac{2}{3}}$$

For Cu the density of atoms is

$$n = 8.5 \times 10^{28} m^{-3}$$

Thus we get the fermi energy equal to

$$E_f = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} = \frac{(6.6\times 10^{-34})^2}{8\times 9.1\times 10^{-31}}\left(\frac{3}{\pi}8.5\times 10^{28}\right)^{-\frac{2}{3}} = 1.1\times 10^{-18} = 6.7eV$$

In kelvin this is equivalent to $6.7eV = 6.4\times 10^4 K$ □

8.4 Homework Four

8.4.1. Derive the virial expansion of the ideal Bose gas by inverting the relation $n\lambda^3 = g_{3/2}(z)$ series to express z in terms of $n\lambda cb$ and the substitute it in the P/KT equation. Using this expression derive the expansion for C_v/Nk valid at high temperature limit.
Solution:
For high temperature $N_0 \ll N$ the relation can be written as

$$n\lambda^3 = g_{3/2}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots$$

To invert the series with usual technique we write the z as a power series in $n\lambda^3$ as

$$z = c_1(n\lambda^3) + c_2(n\lambda^3)^2 + c_3(n\lambda^3)^3 + \dots$$

substuting the value of z into the first series we get

$$n\lambda^3 = \left[c_1(n\lambda^3) + \dots\right] + \left[\frac{\left(c_1(n\lambda^3) + c_2(n\lambda^3)^2\right)^2}{2^{3/2}}\right] + \dots$$

Comparing the coefficients of like powers of λ^3 in both sides we get

$$c_1 = 1; \qquad c_2 + \frac{c_1^2}{2^{-3/2}} = 0 \qquad c_3 + \frac{2c_1c_2}{2^{-3/2}} + \frac{c_1^3}{3^{-3/2}} = 0$$

Writing similarly we get

$$c_1 = 1; c_2 = \frac{-1}{2^{-3/2}} \qquad c_3 = \frac{1}{4} - \frac{1}{3^{-3/2}}$$

Now the expression $\ln \mathcal{Q}$ becomes

$$\frac{PV}{NkT} = \frac{1}{n\lambda^3}\left(z + \frac{z^2}{2^{-5/2}} + \frac{z^3}{3^{-5/2}} + \dots\right)$$

Substuting the value of z from the series in $n\lambda^3$ with the various coefficeints $c_1, c_2 \dots$ we get

$$\frac{PV}{NkT} = \sum_{l=1}^{\infty} a_l \left(\frac{\lambda^3}{v}\right)^{l-1}$$

This is the required virial expansion of the expression. Now for the specific heat at constant volume we have to find out $\frac{\partial U}{\partial T}$, this can be simplified as

$$\frac{C_v}{Nk} = \frac{1}{Nk}\left(\frac{\partial U}{\partial T}\right)_{N,V} = \frac{3}{2}\left[\frac{\partial}{\partial T}\left(\frac{PV}{Nk}\right)\right]_v$$

In similar fashion for the expansion of $g_{5/2}(z)$ we get

$$\frac{C_v}{Nk} = \sum_{l=1}^{\infty} \frac{3}{2}\frac{5-3l}{2} a_l \left(\frac{\lambda^3}{v}\right)^{l-1}$$

Substuting all the coefficient we get

$$\frac{C_v}{Nk} = \frac{3}{2}\left[1 + c_1\left(\frac{\lambda^3}{v}\right) + c_2\left(\frac{\lambda^3}{v}\right)^2 + \dots\right]$$

where teh coefficients are $c1 = 0.088, c_2 = 0.0065, \dots$ This is the expression of specific heat of bose gas correct at high temperature. □

8.4.2. **(Pathria & Beale, 7.3)** Combining equation 7.1.24 and 7.1.26, and making use of the first wo terms of formula (D.9) in Appendix D, show that, as T approaches T, from above the parameter $\alpha(=\ln z)$ of the ideal bose gas assumes the form

$$\alpha = \frac{1}{\pi}\left(\frac{3\zeta(3/2)}{4}\right)^2\left(\frac{T-T_c}{T}\right)^2$$

Solution:
We have from previous problem $n\lambda^3 = g_{3/2}(z)$. But at $\lambda = \lambda_c$ we have $z = 1$. But for $z = 1$

$$g_{3/2}(1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots = \zeta(3/2)$$

Substuting this in the expression for the critical temperature and taking the ratio

$$\frac{T}{T_c} \equiv \left(\frac{\lambda}{\lambda_c}\right)^2 = \left(\frac{g_{3/2}(z)}{\zeta(3/2)}\right)^{-\frac{2}{3}}$$

The expression for $g_{3/2}(z)$ can be expanded in termf os series the series expansion from appendix D.9 can be used to obtain

$$\frac{T}{T_c} = \left(\frac{\zeta 3/2 - 2\sqrt{\pi\alpha} + \dots}{\zeta(3/2)}\right)^{-\frac{2}{3}}$$

Since we have $\alpha \ll 1$ we can make use of binomial expansion of the series

$$(1+x)^n \approx 1 + nx; \qquad x \ll 1$$

Using just the first two terms we get

$$\frac{T}{T_c} \approx 1 + 4\frac{\sqrt{\pi\alpha}}{3\zeta(3/2)}$$

Now, this expression can be simplified further to get

$$4\sqrt{\pi\alpha} = 3\zeta(3/2)\left(\frac{T-T_c}{T_c}\right)$$

Squaring both sides leds to

$$\alpha = \frac{1}{\pi}\left(\frac{3\zeta(3/2)(T-T_c)}{4T}\right)^2$$

This is the required expression. □

8.4.3. Derive in detailed steps the following expression for an ideal Bose gas.

$$\frac{C_v}{Nk} = \frac{15g_{5/2}(z)}{4g_{3/2}(z)} - \frac{9g_{3/2}(z)}{4g_{3/2}(z)}$$

Solution:
For ideal bose gas from 7.1.7 and 7.1.8 we get

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)$$

$$\frac{N - N_0}{V} = \frac{1}{\lambda^3} g_{3/2}(z)$$

At high temperature we can assume that $z \ll 1$ is very small and we can safely ignore N_0. We can take the ratio of these two quantities to get

$$\frac{PV}{NkT} = \frac{g_{5/2}(z)}{g_{3/2}(z)}$$

Also the internal energy can be calculated as

$$U \equiv -\left(\frac{\partial}{\partial \beta} \ln \mathcal{Q}\right)_{z,V} = kT^2 \left(\frac{\partial}{\partial T}\left(\frac{PV}{KT}\right)\right)_{z,v} = \frac{3}{2} kT \frac{V}{\lambda^3} g_{5/2}(z)$$

Now the expression for the specific heat is

$$C_v = \frac{\partial U}{\partial T} = \left[\frac{\partial}{\partial T}\left(\frac{3}{2} T \frac{g_{5/2}(z)}{g_{3/2}(z)}\right)\right]_v$$

Now we can use the recurrance relation for the function $g(z)$ as

$$z \frac{\partial}{\partial z} g_\nu(z) = g_{\nu-1}(z)$$

Also since the function $g_{3/2}(z)$ is proportional to cube root of the square of the temperature we get

$$\left[\frac{\partial}{\partial T} g_{3/2}(z)\right]_v = -\frac{3}{2T} g_{3/2}(z)$$

Combining these two expressions we get

$$\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_v = -\frac{3}{2T} \frac{g_{3/2}(z)}{g_{1/2}(z)}$$

Now carrying out the differentiation of the expression C_v we get

$$C_v = Nk \frac{3}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)} + Nk \frac{\partial}{\partial T}\left(\frac{g_{5/2}(z)}{g_{3/2}(z)}\right) \frac{\partial z}{\partial T}$$

Using the previous expression for $\frac{\partial z}{\partial T}$ and using the product rule in the differentiation we get

$$\frac{C_v}{Nk} = \frac{3}{2}\left[\frac{5}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{3}{2} \frac{g_{3/2}(z)}{g_{1/2}(z)}\right]$$

Simplifying the expression gives

$$\frac{C_v}{Nk} = \frac{15}{4} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)}$$

This is the requred expression for the specific heat of bosons in high temperature limit. □

8.4.4. Prove the following for and atomic Bose gas with spin S

(a) Its density of state is given by:

$$g(E) = 2\pi V(2S+1)\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2}$$

Solution:
If we consider atomic non-interating atomic gas with spin S, then for each momentum state, there are $2S+1$ spin states. Then the grand partition function becomes

$$\mathcal{Q} = \prod_i \mathcal{Q}_i^{2S+1}$$

The grand potential becomes

$$\Phi = -kT\ln\mathcal{Q} = kT(2S+1)\sum_i \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right)$$

Approximating the sum with the integration we get

$$\Phi = kT(2S+1)\int_0^\infty \ln\Big(1+e^{-\beta(\varepsilon-mu)}\Big)g(E)dE$$

Here $g(E)$ is the density of states which cn be simplified for uniformaly distributed particles as

$$g(k)dk = \frac{4\pi k^2(2S+1)dk}{(2\pi/L)^3} = \frac{(2S+1)VK^2dk}{2\pi^2}$$

With volume $V = L^3$ and $E = \frac{h^2k^2}{2m}$ we get

$$g(E)dE = \frac{(2S+1)V\sqrt{E}dE}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$

Using $\hbar = \frac{h}{2\pi}$ and writing the density of states we get

$$g(E) = 2\pi V(2S+1)\sqrt{E}\left(\frac{2m}{h}\right)^{3/2}$$

Which is the required density of states. □

(b) Then show that its Bose-Enistein temperature is given by

$$T_c = \frac{h^2}{2\pi mk}\left[\frac{n}{2.612(2S+1)}\right]^{2/3}$$

Solution:
Now the total number o particles N can be obtained as

$$N = \int_0^\infty \frac{g(E)dE}{e^{\beta(\varepsilon-\mu)}+1}$$

Substuting the density function we get

$$N = \left[2\pi V\left(\frac{2m}{h^2}\right)^{3/2}\right]\int_0^\infty \frac{\sqrt{E}dE}{z^{-1}e^{\beta E}-1}$$

The integrand can be recocnized as the einstein function $g_{3/2}(z)$. so we get

$$N = \frac{(2S+1)V}{\lambda^3} g_{3/2}(z)$$

For $T = T_C$ we can consider z comparable to unity, thus, we have $z = 1$, substuting this we get $g_{3/2}(1) = \zeta(3/2) = 2.612$

$$\frac{n\lambda^3}{2S+1} = \zeta(3/2) = 2.612$$

Making this substution and recocnizing $\lambda = \left[\frac{h^s}{2\pi mkT}\right]^{3/2}$ Rearrainging we get

$$T_c = \frac{h^2 2}{\pi mk}\left[\frac{n}{2.612(2S+1)}\right]$$

where we have made use of $n = \frac{N}{V}$. This gives the expression for the critical temperature of Bose gas. $\square$

Chapter 9

Particle Physics

9.1 Homework One

9.1.1. **(Griffith 1.2)** The mass of Yukawa's meson can be estimated as follows. When two protons in a nucleus exchange a meson (mass m), they must temporarily violate the conservation of energy by an amount mc^2(the rest energy of the meson). The Heisenberg uncertainty principle says that you may 'borrow' an energy ΔE, provided you 'pay it back' in a time Δt given by $\Delta E \Delta t = \frac{\hbar}{2}$ (where $\hbar = h/2\pi$). In this case, we need to borrow $\Delta E = mc^2$ long enough for the meson to make it from one proton to the other. It has to cross the nucleus (size r_0), and it travels, presumably, at some substantial fraction of the peed of light, so , roughly speaking, $\Delta t = \frac{r_0}{c}$. Putting all this together, we have

$$m = \frac{\hbar}{2r_0 c}$$

Using $r_0 = 1 \times 10^{-13}$cm, calculate the mass of Yukuwa's meson. Express your answer in $\frac{MeV}{c^2}$, and compare the observed mass of πon.
Solution:
Given $r_0 = 1 \times 10^{-15} m$, $M = 6.58 \times 10^{-22} MeV s; c = 3 \times 10^8 s$ we can substitute to find the total mass

$$m = \frac{\hbar}{2r_0 c} = \left(\frac{\hbar c}{2r_0} \frac{1}{c^2} \right) = 98.7 \frac{MeV}{c^2}$$

So the predicted mass is $98.7 MeV$, but the real mass of Yukuwa's meson is $138 Mev$ which is off by a factor of about 1.4. □

9.1.2. (a) Members of baryon decuplet typically decay after 1×10^{-23} seconds into a lighter baryon (from the baryon octet) and a meson (from the pseudo-scalar meson octet). Thus for example, $\Delta^{++} \rightarrow p^+ + \pi^+$. List all decay methods of this form for the Δ^-, Σ^{*+} and Ξ^{*-}. Remember that these decays must conserve charge and strangeness(they are strong interactions).
Solution:
The decay has to satisfy the charge conservation and strangeness conservation. The possible decay for each of these are:

$$\begin{aligned} &\Delta^- \rightarrow n + \pi^- \text{ and } \Sigma^- + K^0 \\ &\Sigma^{*+} \rightarrow p + \bar{k}^0; \quad \Sigma^+ + \pi^0; \quad \Sigma^+ + \eta; \quad \pi^0 + \Sigma^0; \quad \Lambda + \pi^+; \quad K^+ + \Xi^0 \\ &\Xi^{*-} \rightarrow \Sigma^0 + K^-; \quad \Xi^- + \pi^0; \quad \Sigma^- + \bar{K}^0; \quad \Lambda + K^-; \quad \Xi^0 + \pi^-; \quad \Xi^- + \eta \end{aligned}$$

These are all the possible decay schemes that preserve charge and strangeness. □

(b) In any decay, there must be sufficient mass in the original particle to cover the masses of the decay products. (There may be more than enough; the extra will be 'soaked up' in the form of kinetic energy int the final state.) Check each of the decay you proposed in part (9.1.2) to see which ones meet this criterion. The others are kinematically forbidden.

Solution:
Each of these decays are two body decays of the form $A \to B + C$, the threshold energies in each can be calculated with

$$E = \frac{M^2 - m_B^2 - m_c^2}{2M_A}$$

Using the mass value of each of these products we find that the only allowed decays are

$$\begin{aligned} \Delta^- &\to \pi^- + n \\ \Sigma^{*+} &\to \Sigma^+ + \pi^0; \qquad \Lambda + \pi^+; \qquad \Sigma^0 + \pi^- \\ \Xi^{*-} &\to \Sigma^0 + \pi^-; \qquad \Xi^- + \pi^0 \end{aligned}$$

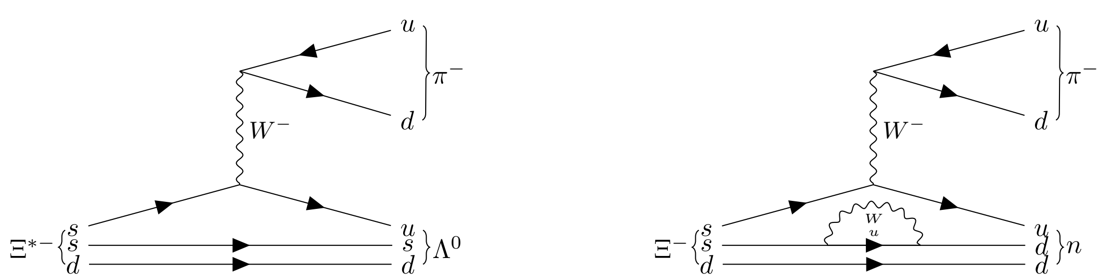
These are the only allowed decays. □

9.1.3. **(Griffith 2.5)**

(a) Which decay do you think would be more likely,

$$\Xi^- \to \Lambda + \pi^- \qquad \text{or} \qquad \Xi^- \to n + \pi^-$$

Solution:
Although the decay $\Xi^- \to n+\pi^-$ is favored kinematically over the decay $\Xi^- \to \Lambda+\pi^-$ strangeness



(a) $\Xi^{*-} \to \pi^- + \Lambda$

(b) $\Xi^{*-} \to \pi^- + n$

Figure 9.1: Feynman diagram for two different decays.

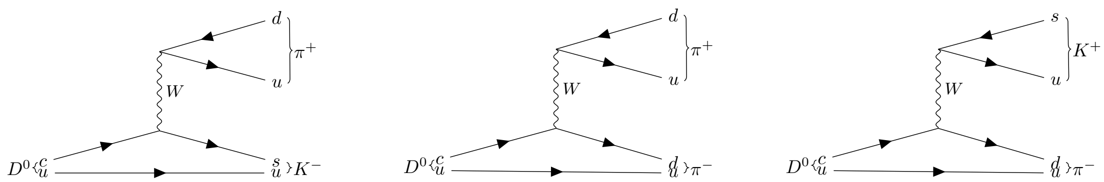
conservation favors the second one. Since the two s quarks have to be conserved (strangeness conservation); an extra W^- is requires. This means there are two extra weak vertices. Higher number of vertices would make the process much more less likely. □

(b) Which decay of $D^0(c\bar{u})$ meson is most likely

$$D^0 \to K^- + \pi^+ \text{ or } D^0 \to \pi^- + \pi^+, \text{ or } D^0 \to K^+ + \pi^-$$

Which is least likely? Draw the Feynman diagrams, explain your answer and check the experimental data.

Solution:
The Feynman diagram for $D^0 \to K^- + \pi^+$ is □
The second decay is more favored because there is no generation cross over in the particle decay. When there is a generation cross over in the decay process it is less favored in the decay although it is allowed kinematically. So the most favored decay process is $D^0 \to \pi^- + \pi^+$.



(a) $D^0 \to \pi^+ + K^-$ (b) $D^0 \to \pi^- + \pi^+$ (c) $D^0 \to \pi^- + K^+$

Figure 9.2: Feynman diagram for three different decay schemes for D^0.

9.1.4. **(Griffith 3.13)** Is p^μ timelike, spacelike, or lightlike for a (real) particle of mass m ? How about a massless particle? How about a virtual particle?
Solution:
To determine the nature of the particles we find the Lorentz scalar for each. Finding $p^2 = p \cdot p = p^\mu p_\mu$ we get

$$p^2 = m^2c^2$$

For a real particle with mass m the quantity $p^2 > 0$ so the particle is timelike. For a massless particle γ the scalar $p^2 = 0$ so this is lightlike. And for virtual particle the nature depends upon the mass as there could be massless and massive virtual particles. □

9.1.5. **(Griffith 3.16)** Particle A(Energy E) hits particle B (at rest), producing $C_1, C_2, \dots A + B \to C_1 + C_2 + \dots C_n$. Calculate the threshold(i.e., minimum E) for this reaction, in terms of various particle masses.
Solution:
In the lab frame lets consider particle A with mass m_A and momentum $\boldsymbol{p}_A$ with energy E strikes a stationary target particle B with mass m_B. The four momentum of A is $p_A^\mu = (E, \boldsymbol{p}_A)$ and the four momentum of B is $p_B^\mu = (m_b, 0)$. The invariant Lorentz scalar in the lab frame is

$$p^2 = (p_A^\mu + p_b^\mu)^2 = (E + m_B, \boldsymbol{p}_A)^2 = E^2 + m_B^2 + 2Em_B - |\boldsymbol{p}_A|^2$$

But for particle A we have $E^2 - |\boldsymbol{p}_A|^2 = m_A^2$ substituting this in above expression we get

$$p^2 = m_A^2 + m_B^2 + 2Em_B$$

Since this Lorentz scalar is invariant in any reference frame we have to have the same value for the p^2 for the final products. For threshold condition the daughter particles are just created so thy do not carry any momentum. Which implies for each particles their momentum $m_n = E_n$ so for each of them the four momentum is $p_n^\mu = (m_n, 0)$. The Lorentz scalar for the final qty is

$$p^2 = (p_1^\mu + p_2^\mu + p_n^\mu)^2 = (m_1 + m_2 + \dots + m_n, 0)^2 = (m_1 + m_2 + \dots + m_n)^2 - 0 = M^2 \text{ (say)}$$

where the symbols M is used to mean the total sum of masses of all daughter particles. Equating the Lorentz scalar we get

$$M^2 = m_A^2 + m_B^2 + 2Em_b \qquad \implies E = \frac{M^2 - m_A^2 + m_B^2}{2m_B}$$

This gives the threshold energy in lab frame of the incoming particle. □

9.1.6. **(Griffith 3.22)** Particle A, at rest, decays into three or more particles: $A \to B + C + D + \dots$

(a) Determine the maximum and minimum energies that B can have in such a decay, in terms of the various masses.

Solution:
The mimimum energy for the outgoing particle is equal to its mass when the produced particle is just created and has no spatial momentum and all other energy is carried away by the other outgoing particles.

$$E_{\text{min}} = m_B$$

The maximum energy is carried by particle B when the particle A decays in such a way that particle B moves in one direction and all other particles move in other direction in unison. Since we would get maximum energy when the other particles do not move relative to each other giving maximum energy, this implies that all other particle move as a single unit of total mass with the sum of their masses. So we can rewrite the decy as

$$A \to B + (C + D \dots) \equiv B + N$$

where the particle N is as it its a single particle with the mass equal to sum of masses of each of the rest of daughter particles.

$$m_N = m_c + m_D + \dots$$

This problem is now like a single particle decaying into two with equal and opposite momentum. In the CM frame the value of Lorentz scalar $p^2 = M_A^2$

$$\begin{aligned} p_A^\mu = p_B^\mu + p_N^\mu \equiv (m_A, 0) &= (E_B, \boldsymbol{p}_B) + (E_N, -\boldsymbol{p}_B) \\ \implies (E_N, -\boldsymbol{p}_B) &= (m_A, 0) - (E_B, \boldsymbol{p}_B) \end{aligned}$$

Squaring both sides and equating

$$\begin{aligned} (E_N, -\boldsymbol{p}_B)^2 &= (m_A, 0)^2 + (E_B, -\boldsymbol{p}_B)^2 - 2(m_A, 0) \cdot (E_B, \boldsymbol{p}_B) \\ E_N^2 - |\boldsymbol{p}_B|^2 &= m_A^2 + E_B^2 - |\boldsymbol{p}_B|^2 - 2m_A E_B \end{aligned}$$

Since we have $m^2 = E^2 - |\boldsymbol{p}|^2$ we get

$$\begin{aligned} m_N^2 &= m_A^2 + m_B^2 - 2m_A E_B \\ 2m_A E_B &= m_A^2 + m_B^2 - m_N^2 \\ E_B &= \frac{m_A^2 + m_B^2 - (m_C + m_D + \dots)^2}{2m_A} \end{aligned}$$

This gives the maximum energy of the particle B. □

(b) Find the maximum and minimum electron energies in muon decay, $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$.
Solution:
The minimum energy of the electron is the mass of electron itself (in natural units of course) so

$$E_{\text{min}} = m_e = 511 keV$$

By above discussion the maximum energy is

$$E_{\text{max}} = \frac{m_\mu^2 + m_{e^-}^2 - (m_{\mu_e} + m_{\bar{\mu}_e})^2}{2m_\mu}$$

Since the neutrinos have very tiny mass (almost massless) we ignore their masses sow we have

$$E_{\text{max}} \approx \frac{105^2 - 0.511^2}{2 \times 105^2} = 52.50 MeV$$

This gives the maximum mass of the outgoing muon. □

9.2 Homework Two

9.2.1. Discuss the possible decay modes of the Ω^- allowed by conservation laws, and show how weak deay is the only remaining choice.
Solution:
There are three decay odes of Ω^-. They are

$$\begin{array}{ccccc} \Omega^- & & \Xi^0 & & \pi^- \\ \text{sss} & \rightarrow & \text{uss} & + & \bar{\text{u}}\text{d} \\ \Omega^- & & \Lambda^0 & & K^- \\ \text{sss} & \rightarrow & \text{uds} & + & \bar{\text{u}}\text{s} \\ \Omega^- & & \Xi^- & & \pi^0 \\ \text{sss} & \rightarrow & \text{dss} & + & \text{u}\bar{\text{u}} \end{array}$$

All of these processes violate the strangeness conservation. So they can't proceed via strong interaction, so weak interaction is the only choice. □

9.2.2. Determine which isospin states the following combination of particles can exist in

(a) $\pi^0\pi^-\pi^0$
Solution:
Using the clebsh Gordan coefficients to write the state composition we get.

$$\begin{aligned} \left|\pi^+\pi^-\pi^0\right\rangle &= |11\rangle\,|1-1\rangle\,|10\rangle \\ |11\rangle\,|1-1\rangle &= \frac{1}{\sqrt{6}}\,|20\rangle + \frac{1}{\sqrt{2}}\,|10\rangle + \frac{1}{\sqrt{3}}\,|00\rangle \\ |20\rangle\,|10\rangle &= \sqrt{\frac{3}{5}}\,|30\rangle - \sqrt{\frac{2}{5}}\,|10\rangle \\ |10\rangle\,|10\rangle &= \sqrt{\frac{2}{3}}\,|20\rangle - \sqrt{\frac{1}{3}}\,|00\rangle \\ |00\rangle\,|10\rangle &= |10\rangle \end{aligned}$$

So the possible iso spin combintaions are $I = \{0, 1, 2, 3\}$ □

(b) $\pi^0\pi^0\pi^0$

$$\begin{aligned} \left|\pi^0\pi^-\pi^0\right\rangle &= |10\rangle\,|10\rangle\,|10\rangle \\ |10\rangle\,|10\rangle &= \sqrt{\frac{2}{3}}\,|20\rangle - \sqrt{\frac{1}{3}}\,|00\rangle \\ |10\rangle\,|00\rangle &= |10\rangle \\ |20\rangle\,|10\rangle &= \sqrt{\frac{3}{5}}\,|30\rangle - \sqrt{\frac{2}{5}}\,|10\rangle \end{aligned}$$

So the possible isospin combintaions are $I = \{1, 3\}$

9.3 Homework Three

9.3.1. **(Griffith 6.6)** The π^0 is a composite object ($u\bar{u}$ and $d\bar{d}$), and so equation 6.23 does not really apply. But lets pretend that the π^0 is a true elementary particle and see how close we came. Unfortunately, we don't know the amplitude $\mathcal{M}$; however it must have the dimensions of mass times velocity, and there is only one mass and one velocity available. Moreover, the emission of each photon introduces a factor of

$\sqrt{\alpha}$ (the fine structure constant) into $\mathcal{M}$, so the amplitude must be proportional to α. On this basis, estimate the lifetime of π^0. Compare the experimental value.

Solution:

The decay rate for a particle decay is given by

$$\Gamma = \frac{S|\boldsymbol{p}|}{8\pi\hbar m_\pi^2}|\mathcal{M}|^2$$

Assuming the decay amplitude is $\mathcal{M} = \alpha m_\pi$ we get

$$\Gamma = \frac{1}{2}\frac{\boldsymbol{p}}{8\pi\hbar m_\pi^2}\left(\alpha m_\pi\right)^2 = \frac{\alpha^2}{16\pi\hbar}|\boldsymbol{p}|$$

The threshold energy of each outgoing photon is

$$E_\gamma = \frac{1}{2}m_\pi$$

We can use the fact that for photon $\left|\boldsymbol{p}_\gamma\right| = E_\gamma$, so the outgoing momentum can be written as

$$\left|\boldsymbol{p}_\gamma\right| = \frac{1}{2}m_\pi$$

Using this in the decay rate expression

$$\Gamma = \frac{\alpha^2 m_\pi}{32\pi\hbar}$$

So the lifetime is given by the reciprocal of decay rate

$$\text{Lifetime}(\tau) = \frac{32\pi\hbar}{\alpha^2 m_\pi}$$

Substuting $\alpha = \frac{1}{137}$ and mass of πon is $135MeV$ we get

$$\tau = \frac{32\pi(6.58\times10^{-22})}{135\cdot\frac{1}{137^2}} = 9.2\times10^{-18}s$$

So the estimated lifetime is 8.4×10^{-18}s. The mean lifetime from the particle data group listing[1] is $(8.30\pm0.19)\times10^{-17}$s, which is off by about an order magnitude.

□

9.3.2. **(Griffith 6.8)** consider th case of elastic scattering, $A + B \to A + B$, in the lab frame, (B initially at rest) assuming the target is so heavy $m_B \gg E_A$ that its recoil is negligible. Determine the differential scattering cross section.

Solution:

In the CM frame, for two body scattering we have the differential scattering cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2\frac{S|\mathcal{M}|^2}{(E_A+E_B)^2}\frac{\left|\boldsymbol{p}_f\right|}{|\boldsymbol{p}_i|}$$

Since the target particle is very heavy, it is essentially at rest after the scattering. So the expression is same for CM frame and lab frame. The energy and momentum of the incoming particle and outgoing particle is essentially the same as the target particle doesn't take any appreciable energy.

$$\left|\boldsymbol{p}_f\right| = |\boldsymbol{p}_i|$$

[1] http://pdg.lbl.gov/2018/listings/rpp2018-list-pi-zero.pdf

For the heavy particle, since it is essentially at rest, the energy is given by

$$E_B^2 = |\boldsymbol{p}_B|^2 + m_B^2$$

Which for $|\boldsymbol{p}_B| \approx 0$ gives

$$E_B = m_B$$

Since given that $E_A \ll m_B$ we can approximate

$$E_A + E_B = E_A + m_B \approx m_B$$

Also the particles are not identical so the factor $S = 1$ Thus the final expression for the scattering is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{m_B^2}$$

This is the required expression for the differential cross section of recoil. □

9.3.3. **(Griffith 6.9)** Consider the collision $1 + 2 \to 3 + 4$ in the lab frame (2 at rest), with particles 3 and 4 massless. Obtain the formula for differential cross section.
Solution:
The expression for the differential cross section for the collision $1 + 2 \to 3 + 4 + \ldots + n$ is given by[2]

$$d\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - \ldots - p_n\right) \times \prod_{i=3}^{n} \frac{1}{2\sqrt{p_i^2 + m_i^2}} \frac{d^3 p_i}{(2\pi)^3}$$

In the lab frame, with particle 2 at rest , we have

$$p_2^2 = |\boldsymbol{p}_2|^2 + m_2^2 = m_2^2$$

Also for the expression under the square root is ,

$$p_1 = (E_1, \boldsymbol{p}_1) \qquad p_2 = (m_2, 0), \Rightarrow p_1 \cdot p_2 = E_1 m_2$$

This gives

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} = \sqrt{E_1^2 m_2^2 - m_1^2 m_2^2} = \sqrt{m_2^2 (E_1^2 - m_1^2)} = m_2 |\boldsymbol{p}_1|$$

Substuting these for $n = 4$ we get,

$$d\sigma = \frac{S\hbar^2}{4 p_1 m_2} \left(\frac{1}{4\pi}\right)^2 \int |\mathcal{M}|^2 \delta \left(E_1 + E_2 - |p_3| - |p_4|\right) \times \delta^3 \left(p_1 + p_2 - p_3 - p_4\right) \frac{d^3 p_3 d^3 p_4}{|p_3||p_4|}$$

the delta function make the p_4 integral trivial as

$$d\sigma = \frac{S\hbar^2}{64\pi^2 |\boldsymbol{p}_1| m_2} \int |\mathcal{M}|^2 \delta \left(E_1 + m_2 - |\boldsymbol{p}_3| - |\boldsymbol{p}_1 - \boldsymbol{p}_3|\right) \frac{d^3 \boldsymbol{p}_3}{|\boldsymbol{p}_3||\boldsymbol{p}_1 - \boldsymbol{p}_3|}$$

Assumming 3 particle scatters off at an angle θ relative to the incident particle 1 we get

$$(\boldsymbol{p}_1 - \boldsymbol{p}_3)^2 = |\boldsymbol{p}_1|^2 + |\boldsymbol{p}_3|^2 - 2|\boldsymbol{p}_1||\boldsymbol{p}_3| \cos\theta$$

Also the volume element in the phase space $d^3\boldsymbol{p}_3$ can be written as

$$d^3 \boldsymbol{p}_3 = |\boldsymbol{p}_3|^2 d|\boldsymbol{p}_3| d\Omega$$

[2] Griffith eq. 6.38

Whore Ω is the solid angle. This enables us to write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2|\boldsymbol{p}_1|m_2}\int\limits_0^\infty |\mathcal{M}|^2\delta\left(E_1 + m_2 - |\boldsymbol{p}_3| - \sqrt{|\boldsymbol{p}_1|^2 + |\boldsymbol{p}_3|^2 - 2|\boldsymbol{p}_1||\boldsymbol{p}_3|\cos\theta}\right)\frac{|\boldsymbol{p}_3|^2 d|\boldsymbol{p}_3|}{|\boldsymbol{p}_3|\sqrt{|\boldsymbol{p}_1|^2 + |\boldsymbol{p}_3|^2 - 2|\boldsymbol{p}_1||\boldsymbol{p}_3|\cos\theta}}$$

At this point all the momentum p are the spatial momentum vectors of each particles (not to confuse with earlier notation p_i to mean four momentum), writing for each $|\boldsymbol{p}_i| = p_i$ and we have this integral in p_3 where p_3 is independent of p_1

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2 p_1 m_2}\int\limits_0^\infty |\mathcal{M}|^2\delta\left(E_1 + m_2 - p_3 - \sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}\right)\frac{p_3^2 dp_3}{p_3\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}}$$

This is no easy integral to work out, but lets try, suppose $x = p_3 + \sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}$ Differentiating this with respect to p_3 we get

$$\frac{\mathrm{d}x}{\mathrm{d}p_3} = 1 + \frac{p_3 - p_1\cos\theta}{\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}} = \frac{\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta} + p_3 - p_1\cos\theta}{\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}} = \frac{x - p_1\cos\theta}{\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}}$$

This gives

$$\frac{dp_3}{\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}} = \frac{dx}{x - p_1\cos\theta}$$

Using this in the integral we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2 p_1 m_2}\int\limits_0^\infty |\mathcal{M}|^2\delta\left(E_1 + m_2 - x\right)\frac{p_3 dx}{x - p_1\cos\theta}$$

This integral however is trivial because of the delta function, as it only picks up the terms for $x = E_1 + m_2$ thus we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2 p_1 m_2}|\mathcal{M}|^2\frac{p_3}{E_1 + m_2 - p_1\cos\theta}$$

This can be simplified to write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar}{8\pi}\right)^2\frac{S|\mathcal{M}|^2 p_3}{m_2 p_1\left(E_1 + m_2 - p_1\cos\theta\right)}$$

This is the expression of the scattering cross section. □

9.4 Homework Four

9.4.1. **(Griffith 7.4)** Show that $u^{(1)}$ $u^{(2)}$ are *orthogonal*, in a sense that $u^{(1)\dagger}u^{(2)} = 0$. Likewise, show that $u^{(3)}$ and $u^{(4)}$ are orthogonal. Are $u^{(1)}$ and $u^{(3)}$ orthogonal?
Solution:
The bispinors $u^{(1)}$ and $u^{(2)}$ are

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x - ip_y}{E+m} \end{pmatrix} \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix}$$

Checking for orthogonality with $u^{(1)\dagger}u^{(2)}$ we get

$$\begin{aligned}
u^{(1)\dagger}u^{(2)} &= \begin{pmatrix}1 & 0 & \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m}\end{pmatrix}\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\-\frac{p_z}{E+m}\end{pmatrix}\\
&= 0+0+\frac{p_z(p_x-ip_y)}{(E+m)^2}-\frac{p_z(p_x-ip_y)}{(E+m)^2}\\
&= 0
\end{aligned}$$

Since the product $u^{(1)\dagger}u^{(2)} = 0$ the two bispinors are orthogonal. Similarly the bispinors $u^{(3)}$ and $u^{(4)}$ are

$$u^{(3)} = \begin{pmatrix}\frac{p_x+ip_y}{E+m}\\-\frac{p_z}{E+m}\\0\\1\end{pmatrix} \qquad u^{(4)} = -\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}$$

Checking for orthogonality with $u^{(3)\dagger}u^{(4)}$ we get

$$\begin{aligned}
u^{(3)\dagger}u^{(4)} &= -\begin{pmatrix}\frac{p_x+ip_y}{E+m} & -\frac{p_z}{E+m} & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}\\
&= \frac{p_z(p_x+ip_y)}{(E+m)^2}-\frac{p_z(p_x+ip_y)}{(E+m)^2}+0+0\\
&= 0
\end{aligned}$$

Since the product $u^{(3)\dagger}u^{(4))} = 0$ the two bispinors are orthogonal.

Now checking for the orthogonality of $u^{(1)}$ and $u^{(3)}$ we get

$$\begin{aligned}
u^{(1)\dagger}u^{(3)} &= \begin{pmatrix}1 & 0 & \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m}\end{pmatrix}\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\-\frac{p_z}{E+m}\\0\\1\end{pmatrix}\\
&= \frac{p_x-ip_y}{E+m}+0+0+\frac{p_x-ip_y}{E+m}\\
&= \frac{2p_x}{E+m}
\end{aligned}$$

Since the product $u^{(1)\dagger}u^{(3))} \neq 0$ the two bispinors are not orthogonal. □

9.4.2. **(Griffith 7.17)**

(a) Express $\gamma^\mu\gamma^\nu$ as a linear combination of $1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5$ and $\sigma^{\mu\nu}$.
Solution:
The quantity $\sigma^{\mu\nu}$ is defined s

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \tag{9.1}$$

Also we know from the anti-commutation relation of the gamma matrices by definition

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}\\
\implies \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\nu &= 2g^{\mu\nu}
\end{aligned} \tag{9.2}$$

Adding (9.1) and (9.2) we get

$$\begin{aligned} 2\gamma^\mu\gamma^\nu &= 2(g^{\mu\nu} - i\sigma^{\mu\nu}) \\ \gamma^\mu\gamma^\nu &= g^{\mu\nu} - i\sigma^{\mu\nu} \end{aligned}$$

Here $g^{\mu\nu}$ is the Mankowski metric and is completely composed of numbers $1, -1$ and 0. So this is the required expression. □

(b) Construct the matrices σ^{12}, σ and σ^{23} and relate them to Σ^1, Σ^2, and Σ^3.
Solution:
By definition

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \qquad \sigma^{12} = \frac{i}{2}\left(\gamma^1\gamma^2 - \gamma^2\gamma^1\right) \tag{9.3}$$

$$\begin{aligned} \left[\gamma^1, \gamma^2\right] &= \gamma^1\gamma^2 - \gamma^2\gamma^1 \\ &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} - \begin{pmatrix} -\sigma_2\sigma_1 & 0 \\ 0 & -\sigma_2\sigma_1 \end{pmatrix} \\ &= \begin{pmatrix} [\sigma_2, \sigma_1] & 0 \\ 0 & [\sigma_2, \sigma_1] \end{pmatrix} = \begin{pmatrix} -2i\sigma_3 & 0 \\ 0 & -2i\sigma_3 \end{pmatrix} \end{aligned}$$

Thus

$$\sigma^{12} = \frac{i}{2}\left[\gamma^1, \gamma^2\right] = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \Sigma^3$$

Similarly

$$\sigma^{13} = \frac{i}{2}\left(\gamma^1\gamma^3 - \gamma^3\gamma^1\right)$$

$$\begin{aligned} \left[\gamma^1, \gamma^3\right] &= \gamma^1\gamma^3 - \gamma^3\gamma^1 \\ &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma_1\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_3 \end{pmatrix} - \begin{pmatrix} -\sigma_3\sigma_1 & 0 \\ 0 & -\sigma_3\sigma_1 \end{pmatrix} \\ &= \begin{pmatrix} [\sigma_3, \sigma_1] & 0 \\ 0 & [\sigma_3, \sigma_1] \end{pmatrix} = \begin{pmatrix} 2i\sigma_2 & 0 \\ 0 & 2i\sigma_2 \end{pmatrix} \end{aligned}$$

Thus

$$\sigma^{13} = \frac{i}{2}\left[\gamma^1, \gamma^3\right] = -\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = -\Sigma^2$$

Similarly

$$\sigma^{23} = \frac{i}{2}\left(\gamma^2\gamma^3 - \gamma^3\gamma^2\right)$$

$$\begin{aligned}
[\gamma^2, \gamma^3] &= \gamma^2\gamma^3 - \gamma^3\gamma^2 \\
&= \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix} - \begin{pmatrix} -\sigma_3\sigma_2 & 0 \\ 0 & -\sigma_3\sigma_2 \end{pmatrix} \\
&= \begin{pmatrix} [\sigma_3, \sigma_2] & 0 \\ 0 & [\sigma_3, \sigma_2] \end{pmatrix} = \begin{pmatrix} -2i\sigma_2 & 0 \\ 0 & -2i\sigma_2 \end{pmatrix}
\end{aligned}$$

Thus

$$\sigma^{23} = \frac{i}{2}[\gamma^2, \gamma^3] = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} = \Sigma^1$$

Here the commutation relation for the Pauli matrices $[\sigma_i, \sigma_j] = \varepsilon_{ijk} 2i\sigma_k$ has been used. This gives us the required relationship. □

9.4.3. **(Griffith 11.4)** As it stands Dirac Lagrangian treats ψ and $\bar{\psi}$ asymmetrically. Some people prefer to deal with them on an equal footing, using the modified Lagrangian

$$\mathcal{L} = \frac{i\hbar c}{2}\left[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right] - (mc^2)\bar{\psi}\psi$$

Apply the Euler-Lagrange equations to this $\mathcal{L}$, and show that you get the Dirac equations and its adjoint.

Solution:

The Euler Lagrange equation is for the Lagrangian density $\mathcal{L}(\partial_\mu\phi_1, \partial_\mu\phi_2, ..., \phi_1, \phi_2, \ldots)$ is

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right) = \frac{\partial\mathcal{L}}{\partial\phi_i}$$

For this modified Lagrangian we get

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) &= \frac{\partial\mathcal{L}}{\partial\bar{\psi}} \\
\partial_\mu\left(\frac{i\hbar c}{2}[-\gamma^\mu\psi]\right) &= \frac{i\hbar c}{2}[\gamma^\mu\partial_\mu\psi] - mc^2\psi \\
\frac{i\hbar c}{2}[-\gamma^\mu\partial_\mu\psi] &= \frac{i\hbar c}{2}[\gamma^\mu\partial_\mu\psi] - mc^2\psi \\
i\hbar(\gamma^\mu\partial_\mu\psi) - mc\psi &= 0 \qquad (9.4)
\end{aligned}$$

Similarly we get the other one with $\bar{\psi}$

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) &= \frac{\partial\mathcal{L}}{\partial\psi} \\
\partial_\mu\left(\frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu]\right) &= \frac{i\hbar c}{2}[-\partial_\mu\bar{\psi}\gamma^\mu] - mc^2\bar{\psi} \\
\frac{i\hbar c}{2}[\partial_\mu\bar{\psi}\gamma^\mu] &= \frac{i\hbar c}{2}[-\partial_\mu\bar{\psi}\gamma^\mu] - mc^2\bar{\psi} \\
i\hbar\left(\partial_\mu\bar{\psi}\gamma^\mu\right) + mc\bar{\psi} &= 0 \qquad (9.5)
\end{aligned}$$

We find out that (9.4) and (9.5) are the Dirac equations and its adjoint. Thus this Lagrangian also gives the same Dirac equations. □

9.4.4. **(Griffith 11.20)** Construct the Lagrangian for ABC theory.
Solution:
Since the ABC model of particles are each scalar particle with spin 0, in free form, each can be described with a Klein-Gordan Lagrangian. So we can obtain the total Lagrangian with free form part of Klein-Gordan and interaction term. The free from Lagrangian is for each particle,if we assume the scalar field ϕ_A, ϕ_B and ϕ_C respectively,

$$\mathcal{L}_A = \frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A - \frac{1}{2}m_A^2\phi_A^2$$
$$\mathcal{L}_B = \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}m_B^2\phi_B^2$$
$$\mathcal{L}_C = \frac{1}{2}\partial_\mu\phi_C\partial^\mu\phi_C - \frac{1}{2}m_C^2\phi_C^2$$

The interaction terms as in the model has the strength of $-ig$. So the interaction term is

$$\mathcal{L}_{\text{int}} = -ig\phi_A\phi_B\phi_C$$

So the final Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A - \frac{1}{2}m_A^2\phi_A^2 + \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}m_B^2\phi_B^2 + \frac{1}{2}\partial_\mu\phi_C\partial^\mu\phi_C - \frac{1}{2}m_C^2\phi_C^2 - ig\phi_A\phi_B\phi_C$$

This is the required Lagrangian densitu for the ABC toy model. □

Chapter 10

Classical Electrodynamics

10.1 Homework One

10.1.1. **(Jackson 1.1)** Use Gauss's theorem to prove the following:

(a) Any excess charge placed on a conductor moust lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)
Solution:
Lets assume that the charge lies inside the volume of the conductor. Making a gaussian surface that lies within a volume of conductor and encloses this assumed charge would imply there is finite flux through this sufrace and hence electric field. But electric field inside a conductor is not possible because otherwise the charges would move and we would no longer have static equilibrium. Thus by contradiction, there can be no charge inside the volume of conductor. □

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to thcarges placed inside it.
Solution:
Lets consider two cases, when there is charge inside the conductor and when there is charge outside the conductor. In the first case if we take a gaussian surface that completely encloses the hollow conductor, by gauss's law we get finite electric field at any arbitrary point outside the hollow conductor. Thus the conductor doesn't shield the outside from electric field.
In the second case, when charge is outside. The flux through the gaussian surface enclosing the conductor is zero as there is no charge inside. Since the electric field outside only induces the charge on the surface of the conductor. There can't be field inisde the hollow conductor. □

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude $\frac{\sigma}{\epsilon_0}$ where σ is the charge density per unit area on the surface.
Solution:
Let us assume an arbitrary gaussian surface parallel and very close to the surface of conductor. In such a case the field at every point on the surface is equal and normal to the the plane of this surface. Using gaussian law for this

$$\int_A \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\epsilon_0}$$
$$EA = \frac{\sigma A}{\epsilon_0}$$
$$\implies E = \frac{\sigma}{\epsilon_0}$$

This shous that the electric field near the surface of conductor is normal to the surface and has magnitude of σ/ϵ_0.

□

10.1.2. **(Jackson 1.3)** Using the Dirac delta functions in the appropriate corrdinates, express the following charge distributions as three-dimesional charge densities $\rho(\boldsymbol{x})$

(a) In spherical coordinates, a charge Q uniformaly distributed over a spherical shell of radious R.
Solution:
Since the total charge Q is uniformly distributed over the surface of shell, and the total surface area of shell is $4\pi R^2$ we have, total surface density given by

$$\frac{Q}{4\pi R^2}$$

Now in the entire space, the only place this surface charge can be found is at the surface of sphere of radious R thus the total charge density over all space becomes

$$\rho(\boldsymbol{x}) = \rho(r,\theta,\phi) = \frac{Q}{4\pi R^2}\delta(r-R)$$

This gives the total charge density over all space if the charge is in spherical shell of radius R . □

(b) In cylindrical coordinates, a charge λ per unit lentgh uniformly distributed over a culindrical surface of radious b.
Solution:
Let us consider a arbitrary length of the cylindrical surface l, with radius b. Now, the total surface area of this arbitrary cylindrical section is $V = 2\pi bl$. The total surface density of charge is for some charge Q is

$$\frac{Q}{2\pi bl}$$

The only place this charge can be found in all of space is for locations where $r = b$ (in cylindrical coordinate). Thus the total charge density over all aspace becomes

$$\rho(\boldsymbol{x}) = \rho(r,\phi,z) = \frac{Q}{2\pi bl}\delta(b-r) = \frac{\lambda}{2\pi b}\delta(b-r)$$

This gives the total charge density over all space if the charge is in cylindrical surface of radius bof radius b provided the linear charge density is λ. □

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.
Solution:
The total area of the circular disc is πR^2. The surface charge density of for this disk is

$$\frac{Q}{\pi R^2}$$

. Now since the disk is negligible thickness the total only place where this charge resides in entire space is where $z = 0$ and $r \leq R$. Thus the total volume charge density over entire space vbecomes

$$\rho(\boldsymbol{x}) = \rho(r,\phi,z) = \frac{Q}{\pi R^2}\delta(z)\mathcal{H}(r-R)$$

where $\mathcal{H}(x) = 1$ if $x < 0, 0$ otherwise.

□

(d) The same as (10.1.2c), but using spherical coordinates.
Solution:
From (10.1.2c), we can make the change of coordinate as $z = r\cos(\theta)$. Substutin that in denta function, and using the property fo denta function

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$$

the total density becomes

$$\rho(\boldsymbol{x}) = \rho(r,\phi,\theta) = \frac{Q}{\pi R^2}\delta(r\cos\theta)\mathcal{H}(r-R) = \frac{Q}{\pi R^2}\frac{1}{r}\delta(\cos\theta)\mathcal{H}(r-R)$$

where $\mathcal{H}(x) = 1$ if $x < 0, 0$ otherwise. □

10.2 Homework Two

10.2.1. **(Jackson 1.6)** A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance. Using Gauss' law, calculate the capacitance of

(a) two large, flat, conducting sheets of area A, separated by small distance d
Solution:
Making a arbitrary Gaussian surface of area S near the surface and parallel to the surface of the large sheet we find that the electric field near the surface is

$$E = \frac{\sigma}{2\epsilon_0}$$

Since between the places both plates have the same field they add up to twice the value. Since the field is uniform between the plates, the potential difference is simply the product of the field and the separation thus we get

$$V = Ed = 2 \times \frac{\sigma}{2\epsilon_0} \times d = \frac{\sigma d}{\epsilon_0}$$

Also the total charge in the entire surface is simply the product the charge density and its area thus we get

$$V = \frac{qd}{A\epsilon_0} \implies C = \frac{q}{V} = \frac{A\epsilon_0}{d}$$

This gives the capacitance of the two large flat plates. □

(b) two concentric conducting spheres with radii a , b ($b > a$);
Solution:
For two conducting sphere, the electric field due to outer sphere in the space between two spheres is zero. The only field is due to the charge on inner conductor. Constructing a spherical Gaussian surface enclosing the inner sphere we find the total field in the region between the two spheres is

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$$

where $a < r < b$ is the distance from the center of the spheres. Now the potential difference between the spheres is the work done on unit charge moving from inner sphere to the out sphere thus we have

$$V = \int_a^b \frac{Q}{4\pi\epsilon_0 r^2}dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab}$$

The capacitance is now simply the ratio of Q and V which is

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}$$

This gives the capacitance of spherical capacitor. □

(c) two concentric conducting cylinders of length L , large compared to their radii a , b ($b > a$).
Solution:
Similar to part (10.2.1b) we get no field inside the inner cylinder and outside the outer cylinder. In the space between the two, only the inner cylinder contributes to the electric field. Again with a cylindrical Gaussian surface bounding the inner cylinder we find that the field in the space between those is

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\epsilon_0} \implies E 2\pi r L = \frac{Q}{\epsilon_0} \implies E = \frac{Q}{2\epsilon_0 \pi L r}$$

where $a < r < b$ is the radial distance from the center of the cylinders. The potential difference now is again the work done on unit charge which is

$$V = \int_a^b \frac{Q}{2\epsilon_0 \pi L r} dr = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

The capacitance is by definition found by the ratio of Q to V;

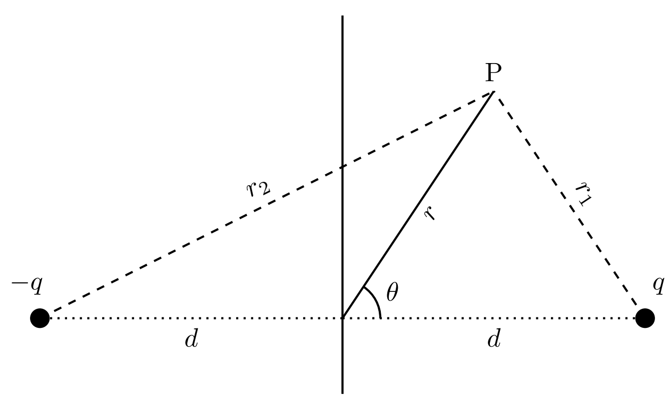
$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

This gives the capacitance of cylindrical capacitor. □

(d) What is the inner diameter of the outer conductor in an air filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m.

10.2.2. **(Jackson 2.1)** A point charge of q is brought to a position a distance d away from an infinite plane conductor held at zero potential . Using the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;
Solution:
The image charge for a point charge near the infinite conductor is behind the plane at a equal distance and the charge is of equal magnitude and opposite sign. Thus the total potential due to the image charge and the point charge (in polar coordinate system) is Using the cosine law, the



different quantities in the given diagram can be written as

$$r_1^2 = r^2 - 2rd\cos\theta + d^2; \qquad \implies \qquad r_1 = \sqrt{1 - 2dr\cos\theta + d^2}$$
$$r_2^2 = r^2 - 2rd\cos(\pi - \theta) + d^2; \qquad \implies \qquad r_2 = \sqrt{1 + 2dr\cos\theta + d^2}$$

The potential at any general point $P(r, \theta)$ is given by

$$\phi(r, \theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Since by gauss's law near a conductor the surface charge density and the normal component of the field are related by the equation

$$E = \frac{\sigma}{\epsilon_0} \implies \sigma = \epsilon_0 E$$

we calculate the gradient of the potential and evaluate at its surface. The gradient is

$$\boldsymbol{E} = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}$$

Evaluating the radial and azimuthal component and evaluating at the surface which corresponds to $\theta = \pi/2$ we get,

$$E_r = \frac{q}{4\pi\epsilon_0}\left[-\frac{-d\cos(\theta) - r}{(d^2 + 2dr\cos(\theta) + r^2)^{\frac{3}{2}}} + \frac{d\cos(\theta) - r}{(d^2 - 2dr\cos(\theta) + r^2)^{\frac{3}{2}}}\right]_{\theta=\frac{\pi}{2}} = 0$$
$$E_\theta = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[-\frac{dr\sin(\theta)}{(d^2 + 2dr\cos(\theta) + r^2)^{\frac{3}{2}}} - \frac{dr\sin(\theta)}{(d^2 - 2dr\cos(\theta) + r^2)^{\frac{3}{2}}}\right]_{\theta=\frac{\pi}{2}} = -\frac{2qd}{4\pi\epsilon_0(d^2 + r^2)^{3/2}}$$

Thus the total charge density is given by

$$\sigma = \epsilon_0 E = -\frac{qd}{2\pi(d^2 + r^2)^{3/2}}$$

This gives the required surface charge density. □

(b) the force between the charge and its image
Solution:
Since the image charge and the point charge are equal and opposite and magnitude and are a total distance $2d$ apart we get the force by columns' law as

$$F = -\frac{q^2}{4\pi\epsilon_0(2d)^2} = -\frac{q^2}{16\pi\epsilon_0 d^2}$$

This is the required force. □

(c) the total force acting on the plane by integrating $\frac{\sigma^2}{2\epsilon_0}$ over the whole plane;
Solution:
Lets assume a small circular area element at a distance r from the center of the circle then the area element is $da = 2\pi r dr$ thus the total area integral over the whole area is

$$F = \int_{r=0}^{\infty} \frac{\sigma^2}{2\epsilon_0} 2\pi r dr = \frac{qd^2}{4\pi\epsilon_0}\int_0^\infty \frac{r}{(r^2 + d^2)^3} dr = \frac{qd^2}{4\pi\epsilon_0}\cdot\frac{1}{4d^4} = \frac{q^2}{16\pi\epsilon_0 d^2}$$

This gives the same force as in the previous part. □

(d) the work necessary to remove the charge q from its position to infinity;
Solution:
With the image charge at d from the surface we have to move the charge from d above the surface to infinity, the total work done is given by

$$W = \int F dz = \int_{z=d}^{\infty} \frac{-q^2}{4\pi\epsilon_0(d+z)^2} dr = -\frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{d+z}\right]_d^{\infty} = -\frac{q^2}{8\pi\epsilon_0 d}$$

This is the required work for the removal of charge to infinity. □

(e) the potential energy between the charge q and its image.
Solution:
The total potential between the charge and image is simply the electric potential of two equal and opposite point charge q at a distance $2d$ thus we get

$$V = -\frac{q^2}{4\pi\epsilon_0(2d)} = \frac{-q^2}{8\pi\epsilon_0 d}$$

This is the potential between the charge. As required, this is exactly the same as we got in (10.2.2d). □

(f) Find the answer to part (10.2.2d) in electron volts for an electron originally one angstrom from the surface.
Solution:
For $d = 1 \times 10^{-10}$ and $q = 1e - 19$ and $\epsilon_0 = 8.85 \times 10^{-12}$ we get

$$V = -\frac{q^2}{8\pi\epsilon d} = 1.15 \times 10^{-18} J = 7.19 eV$$

Thus the potential energy between the charges is $7.19eV$. □

10.2.3. **(Jackson 2.7)** Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity),

(a) Write down the appropriate Green function $G(\boldsymbol{x}, \boldsymbol{x'})$,
Solution:
Let there be a point charge q $\boldsymbol{x'} = (\rho', \phi', z')$. For the potential to be zero at plane $z = 0$ we assume a image charge $-q$ at $(\rho', \phi', -z')$. The green's function is simply the potential due to these point charge at a general location $\boldsymbol{x} = (\rho, \phi, z)$

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{r_1} - \frac{1}{r_2}$$

where r_1 is the distance of general point to the point charge and r_2 is the distance from image charge to the general point. We can calculate the distances as

$$r_1 = \sqrt{(\rho\cos\phi - \rho'\cos\phi)^2 + (z - z')^2 + (\rho\sin\phi - \rho\sin\phi')^2} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z - z')^2}$$

Similarly

$$r_2 = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z + z')^2}$$

Since the choice of coordinate system is arbitrary due to azimuthal symmetry, we can choose $\phi' = 0$ without loss of generality.

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{r_1} - \frac{1}{r_2} = \left[\frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi + (z - z')^2}} - \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi + (z + z')^2}}\right]$$

This is the greens' function. □

(b) if the potential on the place $z = 0$ is specified to be $\Phi = V$ inside a circle of radio's a centered at the origin, and $\Phi = 0$ outside that circle, Find integral expression for the potential ant he point P specified in terms of cylindrical coordinates (ρ, ϕ, z).
Solution:
The integral equation to solve for the potential is

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int_V G(\boldsymbol{x}, \boldsymbol{x'})\rho(x')dV - \frac{1}{4\pi}\oint_s \Phi(\boldsymbol{x'})\frac{\partial G(\boldsymbol{x}, \boldsymbol{x'})}{\partial n}da'$$

Since we don't have charge density inside the volume bounded by the cylinder $\rho(\boldsymbol{x'}) = 0$ thus the only remaining term is the second term. The outward normal on the surface of the cylinder can be calculated. But since the potential at upper infinite plane is zero it has no contribution. Similarly the sidewall of the cylinder do not contribute to the integral because the cylindrical wall have a surface area infinity and the integral goes to zero. Thus the only contribution comes from base of cylinder with radius a . On this face $z' = 0$ so we get

$$\begin{aligned}\left.\frac{\partial G}{\partial n}\right|_{z'=0} &= \left[\frac{1}{2}\frac{2(z-z')}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z-z')^2)^{3/2}} + \frac{1}{2}\frac{2(z+z')}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z+z')^2)^{3/2}}\right]_{z=0}\\ &= \frac{2z}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+z^2)^{3/2}}\end{aligned}$$

Since the potential at that surface is $\Phi(\boldsymbol{x'}) = V$ we get

$$\Phi = \frac{V}{4\pi}\int\limits_{\rho'=0}^{a}\int\limits_{\phi'=0}^{2\pi}\frac{2z}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi+z^2)^{3/2}}\rho' d\phi' d\rho'$$

This is the required integral expression. □

(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Solution:
Solving at $\rho = 0$ we get

$$\Phi(x) = \frac{2Vz}{4\pi}\cdot 2\pi\int\limits_0^a\frac{\rho'}{(\rho'^2+z^2)^{3/2}}d\rho' = Vz\left[\frac{1}{\sqrt{\rho'^2+z^2}}\right]_0^a = Vz\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+a^2}}\right] = V\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]$$

Which is the required expression. □

10.3 Homework Three

10.3.1. **(Jackson 2.13)**

(a) Two halves of a long hollow conducting cylinder of inner radius b and separated by a small length-wise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho, \phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap.
Solution:

The electric potential follows Poisson's equation $\Delta^2\phi = \frac{\rho}{\epsilon_0}$ since, in this particular problem there is no charge in the space, it reduces to Laplace's equation $\Delta^2\phi = 0$. Since the problem entails cylindrical boundary conditions we look for solution of Laplace's equation in cylindrical coordinate system. Also, since the cylinder is long, the potential has no z dependence, we can essentially solve the potential at the bottom plane of the cylinder $z = 0$ and this solution works for every z . So the general solution of Laplace's equation in polar coordinate system is

$$u(\rho,\phi) = (C_0 \ln\rho + D_0) + \sum_n (A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n})$$

Since we expect finite solution at $\rho = 0$, $D_n = 0$ otherwise it $\rho^{-n} = \infty$ which won't satisfy boundary condition. By similar arguments $C_0 = 0$ as the solution has to be finite at $\rho = 0$ but $\ln\rho$ diverges at $\rho = 0$. So the solution reduces to, (absorbing C_n into A_n and B_n)

$$u(\rho,\phi) = D_0 + \sum_n [A_n \cos n\phi + B_n \sin n\phi]\rho^n$$

The boundary condition are, at the edge of the cylinder $\rho = b$, lets choose our coordinate system such that the right half of the cylinder from $\phi = -\frac{\pi}{2}$ to $\phi = \frac{\pi}{2}$ is at potential V_1 and the left half $\phi = \frac{\pi}{2}$ to $\phi = \frac{3\pi}{2}$ is at potential V_2

$$u(b,\phi) = \begin{cases} V_1 & \text{if } -\frac{\pi}{2} < \phi < \frac{\pi}{2} \\ V_2 & \text{if } \frac{\pi}{2} < \phi < \frac{3\pi}{2} \end{cases}$$

Now at the edge of the

$$u(b,\phi) = D_0 + \sum_{n=1}^{\infty} b^n [A_n \cos n\phi + B_n \sin n\phi]$$

Now the constant coefficient D_0 can be easily calculated by integrating both sides as

$$\int_{-\pi/2}^{3\pi/2} u(b,\phi)d\phi = \int_{-\pi/2}^{3\pi/2} D_0 d\phi + \sum_n b^n \int_{-\pi/2}^{3\pi/2} [A_n \cos n\phi + B_n \sin n\phi]d\phi$$

$$\int_{-\pi/2}^{\pi/2} u(b,\phi)d\phi + \int_{\pi/2}^{3\pi/2} u(b,\phi)d\phi = \int_{-\pi/2}^{3\pi/2} D_0 d\phi + \sum_{n=1}^{\infty} b^n \left\{ A_n \cancelto{0}{\int_{-\pi/2}^{3\pi/2} \cos n\phi d\phi} + B_n \cancelto{0}{\int_{-\pi/2}^{3\pi/2} \sin n\phi d\phi} \right\}$$

$$\begin{aligned} V_1\pi + V_2\pi &= D_0 2\pi + 0 \\ D_0 &= \frac{V_1 + V_2}{2} \end{aligned}$$

Again the coefficients B_n and A_n can be calculated by using the fact that $\{\sin\phi\}_n$ and $\{\cos\phi\}_n$ form an orthogonal set of function for integer set of n. Integrating the above expression by multiplying by $\sin m\phi$ on both sides gives

$$\begin{aligned} \int_{-\pi/2}^{3\pi/2} u(b,\phi)\sin m\phi d\phi &= \sum_n b^n \left\{ A_n \cancelto{0}{\int_{-\pi/2}^{3\pi/2} \cos n\phi \sin m\phi d\phi} + B_n \int_{-\pi/2}^{3\pi/2} \sin n\phi \sin m\phi d\phi \right\} \\ &= \sum_n B_n b^n \frac{2\pi}{2}\delta_{mn} = B_m b^m \pi \\ \Rightarrow \quad B_m &= \frac{1}{\pi b^m} \int_{-\pi/2}^{3\pi/2} u(b,\phi)\sin m\phi d\phi \end{aligned}$$

Similarly the coefficients A_m can be calculated as

$$A_m = \frac{1}{\pi b^m} \int_{-\pi/2}^{3\pi/2} u(b,\phi)\cos m\phi d\phi$$

Since in the given problem $u(a,\phi)$ has different values for different ϕ we get

$$\begin{aligned}
A_m &= \frac{1}{\pi b^m}\left[\int_{-\pi/2}^{\pi/2} u(b,\phi)\cos m\phi d\phi + \int_{\pi/2}^{3\pi/2} u(b,\phi)\cos m\phi d\phi\right] \\
&= \frac{1}{\pi b^m}\left[V_1 \int_{-\pi/2}^{\pi/2} \cos m\phi d\phi + V_2 \int_{\pi/2}^{3\pi/2} \cos m\phi d\phi\right] \\
&= \frac{1}{\pi b^m}\left[V_1\left(\frac{1-(-1)^m}{m}\right) + V_2\left(\frac{(-1)^m - 1}{m}\right)\right] \\
&= \frac{1}{\pi b^m}\left[(V_1 - V_2)\left(\frac{1-(-1)^m}{m}\right)\right]
\end{aligned}$$

Wirking out the integral for B_m leads to $B_m = 0$ for all m. So the final solution becomes

$$\begin{aligned}
u(\rho,\phi) &= \frac{V_1+V_2}{2} + \sum_{n=1}^{\infty} \rho^n \frac{1}{\pi b^n}(V_2 - V_2)\frac{1-(-1)^n}{n}\cos n\phi \\
&= \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \frac{1-(-1)^n}{n}\cos n\phi
\end{aligned}$$

clearly the sum term is zero for even n, for odd n the expression is just 2/n. The closed form of the sum gives the required expression

$$u(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\arctan\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

This gives the potential everywhere inside the cylinder. □

(b) Calculate the surface-charge density of each half of the cylinder.
Solution:
The charge density can be simply found by finding the normal component of electric field at the surface.

$$\sigma(\phi) = \epsilon_0 \frac{\partial u(\rho,\phi)}{\partial\rho}\bigg|_{\rho=b} = \epsilon_0 \frac{V_1-V_2}{\pi}\frac{\partial}{\partial\rho}\arctan\left(\frac{2\rho\cos\phi}{b^2-\rho^2}\right)$$

This derivative was evaluated by using sympy to obtain

$$\sigma(\phi) = \epsilon_0\frac{V_1-V_2}{\pi}\frac{4b^2\cdot 2b\cdot\cos\phi}{2b^4+2b^4\cos 2\phi} = \epsilon_0\frac{V_1-V_2}{\pi}\frac{2\cos\phi}{b(1+\cos 2\phi)} = \epsilon_0\frac{V_1-V_2}{\pi b\cos\phi}$$

For each halves we have the condition for ϕ. Subsisting the value of ϕ for each halves gives the charge density of each half. □

10.3.2. **(Jackson 2.15)**

(a) Show that the green function $G(x, y, x\prime, y\prime)$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \leq x \leq 1$, $0 \leq y \leq 1$, has an expansion

$$G(x, y, x\prime, y\prime) = 2\sum_{n=1}^{\infty} g_n(y, y\prime)\sin(n\pi x)\sin(n\pi x\prime)$$

where $g_n(y, y\prime)$ satisfies

$$\left(\frac{\partial^2}{\partial y\prime^2} - n^2\pi^2\right) g_n(y, y\prime) = -4\pi\delta(y\prime - y) \quad \text{and} \quad g_n(y, 0) = g_n(y, 1) = 0$$

Solution:
The green's function solution to non homogeneous differential equation $\mathcal{L}h(x) = f(t)$ is a solution to homogeneous part of the differential equation with the source part replaced as delta function $\mathcal{L}h(x) = \delta(t - \xi)$. The obtained solution is $G(t, \xi)$, i.e., $\mathcal{L}G(t, \xi) = \delta(t - \xi)$. This solution corresponds to the homogeneous part only as it is independent of any source term $f(t)$. Let $G(t, \xi)$ be the solution to the differential equitation with the inhomogeneous part replaced by delta function $\delta(t - \xi)$. The green's function solution to Laplace's equation is then:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^y}{\partial y^y}\right) G(x, y; x\prime, y\prime) = -4\pi\delta(x\prime - x)\delta(y\prime - y)$$

Since we have boundary condition that $G(x' = 0) = 0$ and $G(x' = 1) = 0$ we take odd function fourier expansion of the Green's function

$$G(x, y; x\prime, y\prime) = \sum_{n=1}^{\infty} f_n(x, y; y\prime)\sin(n\pi x\prime) \tag{10.1}$$

Using this expression in the Laplace's equation we obtain

$$\sum_{n=1}^{\infty}\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) f_n(x, y, y\prime)\sin(n\pi x\prime) = -4\pi\delta(x\prime - x)\delta(y\prime - y) \tag{10.2}$$

Completeness of the orthogonal functions $\sin(n\pi x)$ allows us to write the delta function as

$$\delta(x - x\prime) = \sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x\prime)$$

Replacing this expression in (10.2) we obtain

$$\sum_{n=1}^{\infty}\left(\frac{\partial^2}{\partial y\prime^2} - n^2\pi^2\right) f_n(x, y; y\prime)\sin(n\pi x\prime) = -4\pi\delta(y\prime - y)\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x\prime) \tag{10.3}$$

Comparing the function behavior of parameter x on LHS and RHS of (10.3) we obtain that the function f_n is sinusoidal. Separating out the y part of the expression into other function g_n we get

$$f_n(x, y; y\prime) = g_n(y, y\prime)\sin(4\pi x)$$

Now we can substitute this back into our green's function G in (10.1) we get

$$G(x, y; x\prime, y\prime) = \sum_{n=1}^{\infty} g_n(y, y\prime)\sin(n\pi x)\sin(n\pi x\prime)$$

Substituting this expression back to (10.2) we obtain

$$\left(\frac{\partial^2}{\partial y\prime^2} - n^2\pi^2\right) g_n(y, y\prime) = -4\pi\delta(y\prime - y) \tag{10.4}$$

This expression g_n also has to satisfy the boundary conditions as the complete greens function G so we have $g_n(y, 0) = 0$ and $g_n(y, 1) = 1$ as required. □

(b) Taking for $g_n(y, y\prime)$ appropriate linear combinations of $\sinh(n\pi y\prime)$ and $\cosh(n\pi y\prime)$ in the two regions $y\prime < y$ and $y\prime > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x, y; x\prime, y) = 0\sum_{n=1}^{\infty} \frac{1}{n\sinh(n\pi)} \sin(n\pi x)\sinh(n\pi x\prime)\sinh(n\pi y_<)\sinh(n\pi(1 - y_>))$$

where $y_<(y_>)$ is the smaller (larger) of y and $y\prime$.

Solution:

Now that we have the general expression for the green's function (10.4) we can divide the region into parts with $x\prime > x$ and $x\prime < x$, since in each of these cases, the source term in the differential equation is zero as the delta function is zero there $\delta(x' - x) = 0$ if $x' \neq x$ so we get

$$g_n(y, y\prime) = \begin{cases} g_< \equiv a_< \sinh(n\pi y\prime) + b_< \cosh(n\pi y\prime) & \text{if } y\prime < y \\ g_> \equiv a_> \sinh(n\pi y\prime) + b_> \cosh(n\pi y\prime) & \text{if } y\prime > y \end{cases}$$

Finding this function is, down to finding the unknown coefficients $a_<, a_>, b_<, b_>$. Applying the boundary condition $g_n(y, 0) = 0 = g_n(y, 1)$ we get

$$g_> = g_< \qquad \frac{\partial}{\partial y\prime} g_< = \frac{\partial}{\partial y\prime} g_> + 4\pi \qquad \text{if } y\prime > y$$

Now the boundary condition such that $g_>(y' = 1) = 0$ and $g_<(y' = 0) = 0$ suggests sinh functions suit the boundary condition than sin. Thus we get

$$g_n(y, y\prime) = \begin{cases} a_< \sinh(n\pi y\prime) & \text{if } y\prime < y \\ a_>[\sinh(n\pi y\prime) - \tanh(n\pi)\cosh(n\pi y\prime)] & \text{if } y\prime > y \end{cases}$$

Continuity requires that the function match at $y = y\prime$ so we have

$$a_< \sinh(n\pi y\prime) = a_>[\sinh(n\pi y\prime) - \tanh(n\pi y\prime)] \tag{10.5}$$

and the jump discontinuity of greens function require

$$\frac{\partial}{\partial y\prime} g_n(y_<) - \frac{\partial}{\partial y\prime} g_n(y_>) = 1 \tag{10.6}$$

The equations (10.5) and (10.6) give system of equation which can be solved as

$$\begin{pmatrix} \sinh(n\pi y) & -\sinh(n\pi y) + \tanh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi y) & -\cosh(n\pi y) + \tanh(n\pi)\cosh(n\pi y) \end{pmatrix} \begin{pmatrix} a_< \\ a_> \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{n} \end{pmatrix} =$$

Again solving this with simply gives

$$\begin{pmatrix} a_< \\ a_> \end{pmatrix} = -\frac{4}{n\sinh(n\pi)} \begin{pmatrix} \cosh(n\pi)\sinh(n\pi y) - \sinh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi)\sinh(n\pi y) \end{pmatrix}$$

Substituting the coefficients we get

$$g_n(y, y\prime) = \frac{4}{n\sinh(n\pi)} \times \begin{cases} \sinh(n\pi y\prime)[\sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)] & \text{if } y\prime < y \\ \sinh(n\pi y)[\sinh(n\pi)\cosh(n\pi y\prime) - \cosh(n\pi)\sinh(n\pi y\prime)] & \text{if } y\prime > y \end{cases}$$

As required the greens function is symmetric in its parameters. The symmetry is such that the expression looks exactly same if the parameter are exchanged. If we denote $y_<$ to be the minimum of $y$and $y\prime$ and similarly for $y_>$ we can write the above expression in a compact way as

$$g_n(y, y\prime) = \frac{4}{n\sinh(n\pi)} \sinh(n\pi y_<)\sinh(n\pi(1 - y_>))$$

Subsisting this to the greens function solution we get

$$G(x, y, x\prime, y\prime) = \sum_{n=1}^{\infty} \frac{8}{n \sinh(n\pi)} \sin(n\pi x) \sin(n\pi x\prime) \sinh(n\pi y_<) \sinh(n\pi(1 - y_>))$$

This is the required expression for the green's function. □

10.4 Homework Four

10.4.1. **(Jackson 3.1)** Two concentric spheres have radii a , b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and teh lower hemisphere of the outer sphere aare maintained at potential V . The other hemispheres are at zero potential.

Determinte the potential int eh region $a \leq r \leq b$ as the series in Legendre polynomials. Include terms at least upt to $l = 4$. Check your solution against known results in the limiting chases $b \to \infty$, and $a \to 0$.

Solution:

The general solution to Laplace's equation in spherical coordinate system is

$$u(r, \theta, \phi) = [Ar^l + Br^{-(l+1)}][C \cos m\phi + D \sin m\phi][EP_l^m(\cos\theta) + FQ_l^m(\cos\theta)]$$

Since there is azimuthal symmetry the value of $m = 0$. The potential is finite at both the poles, but the associated Legendre function of second kind $Q_l^m(x)$ diverges at $x = \pm 1$, which corresponds to poles, so we require $F = 0$. Absorbing constant C and F into A_k and B_k, the general solution reduces to

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{10.7}$$

Here the function $P_l^0(x) = P_l(x)$ is the Legendre polynomial.

Multiplying both sides by $P_k(\cos\theta)$ and integrating with respect to $d\cos\theta$ from -1 to 1 we get

$$\begin{aligned}
\int_{-1}^{1} u(r, \theta, \phi) P_k(\cos\theta) d\cos\theta &= \int_{-1}^{1} \sum_{l=0}^{\infty} [A_k r^l + B_k r^{-(l+1)}] P_l(\cos\theta) P_k(\cos\theta) d\cos\theta \\
&= \sum_{l=0}^{\infty} \left[A_k r^l + B_k r^{-(l+1)}\right] \frac{2}{2l+1} \delta_{lk} \\
&= \left[A_k r^k + B_k r^{-(k+1)}\right] \frac{2}{2k+1}
\end{aligned}$$

Now evaluating the integral for $r = a$ and $r = b$ respectively.

For $r = a$

$$
\begin{aligned}
\frac{2}{2k+1}[A_k a^k + B_k a^{-(k+1)}] &= \int_{-1}^{1} u(a,\theta,\phi) P_k(\cos\theta) d\cos\theta \\
&= \int_{-1}^{0} 0 \cdot P_k(\cos\theta) d\cos\theta + \int_{0}^{1} V P_k(\cos\theta) d\cos\theta \\
&= \int_{0}^{1} V P_k(x) dx \\
&= \frac{V}{2k+1}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right]
\end{aligned}
$$

Which implies

$$
A_k a^k + B_k a^{-(k+1)} = \frac{V}{2}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right] = \beta \text{ say} \tag{10.8}
$$

Again doing this for $r = b$ we get

$$
\begin{aligned}
\frac{2}{2k+1}[A_k b^k + B_k b^{-(k+1)}] &= \int_{-1}^{1} u(b,\theta,\phi) P_k(\cos\theta) d\cos\theta \\
&= \int_{-1}^{0} V \cdot P_k(\cos\theta) d\cos\theta + \int_{0}^{1} 0 \cdot P_k(\cos\theta) d\cos\theta \\
&= \int_{0}^{1} V P_k(-x) dx \\
&= \int_{0}^{1} V(-1)^k P_k(x) dx \\
&= \frac{V(-1)^k}{2k+1}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right]
\end{aligned}
$$

Which implies

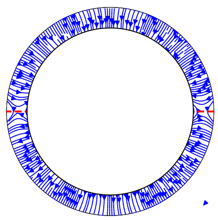
$$
A_k b^k + B_k b^{-(k+1)} = \frac{V(-1)^k}{2}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right] = (-1)^k\beta
$$

So the two linear equations are

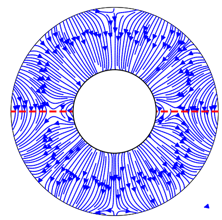
$$
\begin{aligned}
A_k a^k + B_k a^{-(k+1)} &= \beta \\
A_k b^k + B_k b^{-(k+1)} &= (-1)^k\beta
\end{aligned}
$$

We can cast these two equation of unknowns A_k and B_k into matrix equation as

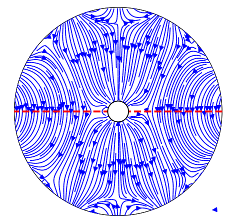
$$
\begin{bmatrix} a^k & a^{-(k+1)} \\ b^k & b^{-(k+1)} \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix} = \begin{bmatrix} \beta \\ (-1)^k\beta \end{bmatrix} \tag{10.9}
$$



(a) Eletric field lines with b/a = 1.25



(b) Eletric field lines with b/a = 2.5



(c) Eletric field lines with b/a = 10

Solving the matrix equation we get the matrix

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \begin{bmatrix} \frac{\beta\left(a^{k+1}-(-1)^k b^{k+1}\right)}{a^{2k+1}-b^{2k+1}} \\ \frac{\beta(ab)^{k+1}\left(a^k-(-1)^k b^k\right)}{a^{2k+1}-b^{2k+1}} \end{bmatrix}$$

Substuting the value of β from (10.8) we get

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \frac{V}{2} \begin{bmatrix} \frac{\Gamma\left(\frac{1}{2}\right)\left(a^{k+1}+(-b)^{k+1}\right)\left(\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)-\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\right)}{(a^{2k+1}-b^{2k+1})\Gamma\left(-\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)} \\ -\frac{\Gamma\left(\frac{1}{2}\right)(ab)^{k+1}\left(b^k-(-a)^k\right)\left(\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)-\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\right)}{(a^{2k+1}-b^{2k+1})\Gamma\left(-\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)} \end{bmatrix}$$

The coefficients are all zeros for all even $k \geq 2$.

$$\begin{bmatrix} A_{2m} \\ B_{2m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \forall m \in \mathbb{Z}_+;$$

The first few odd of this coefficient are

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = V \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}; \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = V \begin{bmatrix} \frac{3(a^2+b^2)}{4(a^3-b^3)} \\ \frac{-3a^2b^2(a+b)}{4(a^3-b^3)} \end{bmatrix}; \begin{bmatrix} A_3 \\ B_3 \end{bmatrix} = V \begin{bmatrix} \frac{-7(a^4+b^4)}{16(a^7-b^7)} \\ \frac{7a^4b^4(a^3+b^3)}{16(a^7-b^7)} \end{bmatrix}; \begin{bmatrix} A_5 \\ B_5 \end{bmatrix} = V \begin{bmatrix} \frac{11(a^6+b^6)}{32(a^{11}-b^{11})} \\ \frac{-11a^6b^6(a^5+b^5)}{32(a^{11}-b^{11})} \end{bmatrix}$$

Substuting these coefficients in (10.7) we obtain the potential in with this boundary condition.

$$u(r,\theta,\phi) = V\left[\frac{1}{2} + \frac{3}{4}\left(\frac{r(a^2+b^2)}{a^3-b^3} - \frac{a^2b^2(a+b)}{r^2(a^3-b^3)}\right)P_1(\cos\theta) + \frac{7}{16}\left(-\frac{r^3(a^4+b^4)}{a^7-b^7} + \frac{a^4b^4(a^3+b^3)}{r^4(a^7-b^7)}\right)P_3(\cos\theta) \right.$$
$$\left. + \frac{11}{32}\left(\frac{r^5(a^6+b^6)}{a^{11}-b^{11}} - \frac{a^6b^6(a^5+b^5)}{r^6(a^{11}-b^{11})}\right)P_5(\cos\theta) + \frac{75}{256}\left(-\frac{r^7(a^8+b^8)}{a^{15}-b^{15}} + \frac{a^8b^8(a^7+b^7)}{r^8(a^{15}-b^{15})}\right)P_7(\cos\theta) + \cdots\right]$$

This is the required potential in the region $a \leq r \leq b$. In the limit $b \to \infty$ we have

$$u(r,\theta,\phi) = V\left[\frac{1}{2} + \frac{3}{4}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{16}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{32}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) - \frac{75}{256}\left(\frac{a}{r}\right)^8 P_7(\cos\theta) + \ldots\right]$$

In the limit $b \to \infty$ the problem is they potential of splatted sphere everywhere outside the sphere. And the above expression matches the expected result. In the limit $a = 0$, which corresponds to the potential inside the sphere inside the splitted potential.

$$u(r,\theta,\phi) = V\left[\frac{1}{2} - \frac{3}{4}\frac{r}{b}P_1(\cos\theta) + \frac{7}{16}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) - \frac{11}{32}\left(\frac{r}{b}\right)^5 P_5(\cos\theta) + \frac{75}{256}\left(\frac{r}{b}\right)^7 P_7(\cos\theta) + \ldots\right]$$

Which also matches our expectation. □

10.5 Homework Five

10.5.1. **(Jackson 3.6)** Two point charges q and $-q$ are located on the z axis at $z = a$ and $z = -a$ respectively

(a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$.

Solution:

Le the position vector of point charges $+q$ and $-q$ be $\boldsymbol{r}_1(a, 0, \phi)$ and $\boldsymbol{r}_2(-a, \pi, \phi)$ respectively. Any point with position vector $\boldsymbol{r}$ will have potential given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_2|}\right]$$

If the angle between two position vectors $\boldsymbol{r}$ and $\boldsymbol{r'}$ is γ, a function of this form, with the help of cosine law, can be written as

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r'}|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \left\{\begin{array}{ll} \frac{1}{r'\sqrt{1+\left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\gamma}} & \text{if } r' \geq r \\ \frac{1}{r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\gamma}} & \text{if } r' < r \end{array}\right\} = \sum_{n=0}^{\infty}\left(\frac{r_<^n}{r_>^{n+1}}\right)P_n(\cos\gamma)$$

Here, $r_> = \max(r, r')$ and $r_< = \min(r, r')$. Also the generating function expansion of legendre polynomials has been used

$$\forall\, t < 1 : \frac{1}{\sqrt{1 + t^2 - 2tx}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

By using the addition theorem for the legendre polynomials we can write

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta_1, \phi_1)^* Y_l^m(\theta, \phi)$$

So we can write the expression

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} = \sum_{l=0}^{\infty}\frac{r^n}{r_1^{n+1}}\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta_1, \phi_1)^* Y_l^m(\theta, \phi)$$

Since we have $|\boldsymbol{r}_1| = |\boldsymbol{r}_2| (= a)$, we can generalize $r_> =$ $\max(r, a)$ and $r_< = \min(r, a)$. So the potnetial expression becomes

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\sum_{l=o}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left\{Y_l^m(\theta_1, \phi_1)^* Y_l^m(\theta, \phi) - Y_l^m(\theta_2, \phi_2)^* Y_l^m(\theta, \phi)\right\} \\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=o}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left[\ Y_l^m(0, \phi)^* - Y_l^m(\pi, \phi)^*\right] Y_l^m(\theta, \phi)\end{aligned}$$

For the given problem $\theta_1 = 0, \theta_2 = \pi$. But

$$\begin{aligned}&\forall m \neq 0 : Y_l^m(0, \phi) = 0 \quad \wedge \quad Y_l^0(0, \phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(1) \quad \Rightarrow Y_l^m(0, \phi) = \sqrt{\frac{2l+1}{4\pi}}\delta_{m,0} \\ &\forall m \neq 0 : Y_l^m(\pi, \phi) = 0 \quad \wedge \quad Y_l^0(\pi, \phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(-1) \quad \Rightarrow Y_l^m(\pi, \phi) = \sqrt{\frac{2l+1}{4\pi}}(-1)^l\delta_{m,0}\end{aligned}$$

Substuting these we get

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\frac{r_<^l}{r_>^{l+1}}\left[\quad(1-(-1)^l)\delta_{m,0}\right]Y_l^m(\theta,\phi)\\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}}\frac{r_<^l}{r_>^{l+1}}(1-(-1)^l)Y_l^0(\theta,\phi)\end{aligned}$$

Since $Y_l^0(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$ and $\forall k\in\mathbb{N}:(1-(-1)^{2k}=0)\wedge(1-(-1)^{2k+1}=2)$, we get

$$\forall k\in\mathbb{N}:\Phi=\frac{2q}{4\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{r_<^{2k+1}}{r_>^{2k+2}}\right)P_{2k+1}(\cos\theta)=\begin{cases}\frac{q}{2\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{r^{2k+1}}{a^{2k+2}}\right)P_{2k+1}(\cos\theta) & \text{if } r\le a\\ \frac{q}{2\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{a^{2k+1}}{r^{2k+2}}\right)P_{2k+1}(\cos\theta) & \text{if } r>a\end{cases}$$

This is the required expression for teh potential due to this dipole. □

(b) Keeping the product $qa=p/2$ constant, take the limit of $a\to 0$ and find the potential for $r\neq 0$. This is by definition a dipole along the z axis and its potential.

Solution:

In the limit $a\to 0$ we have $r>a$ so we get

$$\begin{aligned}\Phi &= \lim_{a\to 0}\frac{q}{2\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{a^{2k+1}}{r^{2k+2}}\right)P_{2k+1}(\cos\theta)\\ &= \lim_{a\to 0}\frac{qa}{2\pi\epsilon_0}\left(\frac{1}{r^2}P_1(\cos\theta)+\frac{a^2}{r^3}P_3(\cos\theta)+\dots\right)\\ &= \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\end{aligned}$$

This is the required expression for potential due to a dipole. □

(c) suppose now that the dipole in (10.5.1b) is surrounded by a grounded spherical shell of radius b concentric with the origin. By linear superposition find the potential everywhere inside the shell.

Solution:

Since the grounded sphere attains charge due to induction of the dipole inside it. It creates its own electric potential inside the sphere which follows Laplace's equation.The general solution to Laplace's equation in spherical coordinate system is

$$u(r,\theta,\phi)=[Ar^l+Br^{-(l+1)}][C\cos m\phi+D\sin m\phi][EP_l^m(\cos\theta)+FQ_l^m(\cos\theta)]$$

Since there is azimuthal symmetry the value of $m=0$. The potential is finite at both the poles, but the associated Legendre function of second kind $Q_l^m(x)$ diverges at $x=\pm 1$, which corresponds to poles, so we require $F=0$. Also since the potential is finite at $r=0$ we require $B=0$ Absorbing constant E into A_k, the general solution reduces to

$$u(r,\theta,\phi)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) \tag{10.10}$$

Here the function $P_l^0(x)=P_l(x)$ is the Legendre polynomial. By superposition principle the totential inside the sphere of radius b must be potential due to the induced charge in sphere and the potential by dipole. So potential everywhere inside the sphere is

$$\Phi'=\Phi+\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)=\frac{p}{4\pi\epsilon_0}\frac{P_1(\cos\theta)}{r^2}+\sum_{l=0}^{\infty}Ar^lP_l(\cos\theta)$$

But we require $\Phi' = 0$ at $r = b$.

$$\sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta) = -\frac{q}{4\pi\epsilon_0}\frac{P_1(\cos\theta)}{b^2}$$

Since $\{P_l(x); l \in \mathbb{N}\}$ form a set of orthogonal functions the coefficient of $P_l(x)$ on either side of equation must be equal for this equation to be identity, thus we get

$$A_1 b = -\frac{q}{4\pi\epsilon_0}\frac{1}{b^2} \implies A_1 = -\frac{q}{4\pi\epsilon_0}\frac{1}{b^3}; \qquad A_l b^l = 0, \implies A_l = 0; \forall\, l \neq 1$$

Using the value of A_l in (10.10) we get

$$\Phi' = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} - \frac{q}{4\pi\epsilon_0 b^3} r\cos\theta = \frac{1}{4\pi\epsilon_0}\left[\frac{p}{r^2} - \frac{r}{b^3}\right]\cos\theta$$

This is the required potential everywhere inside the sphere □

10.5.2. **(Jackson 4.1)** Try to obtain results for the non vanishing moments valid or all l, but in each case find the first two sets of non vanishing moments at the very least. Calculate the multipole moments q_{lm} of the charge distributions shown

(a)

Solution:
The charge density can be written as

$$\rho(\boldsymbol{x}) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\left[\delta(\phi) + \delta\left(\phi + \frac{\pi}{2}\right) - \delta(\phi - \pi)\delta\left(\phi + \frac{3\pi}{2}\right)\right]$$

Since all the charges are in plane $\theta = \frac{\pi}{2}$ so $\cos\theta = 0$. The multipole moments are given by

$$\begin{aligned} q_{lm} &= \int r^l Y_l^m(\theta,\phi)\rho(\boldsymbol{x}) d^3x \\ &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} q a^l P_l^m(0)\left[1 + e^{-im\pi/2} - e^{-im\pi} - e^{-im3\pi/2}\right] \end{aligned}$$

Sicne $P_l^m(0) = 0$ for all even m we can write $m = 2k+1; k \in \mathbb{N}$

$$\begin{aligned} q_l^{2k+1} &= 2qa^l\left[1 - i(-1)^k\right]\sqrt{\frac{2l+1}{4\pi}\frac{(l-(2k+1))!}{(l+(2k+1))!}} P_l^{2k+1}(0) \\ &= 2qa^l\left[1 - i(-1)^k\right] Y_l^{2k+1}\left(\frac{\pi}{2}, 0\right) \end{aligned}$$

This vanishes for all even l thus the values for odd l and m are

$$\begin{aligned} q_{1,1} &= -q_{1,-1}^* = -2qa(1-i)\sqrt{\frac{3}{8\pi}} \\ q_{3,3} &= -q_{3,-3}^* = 2qa^3(1+i)\sqrt{\frac{35}{4\pi}} \\ q_{3,1} &= -q_{3,-1}^* = 2qa^3(1-i)\frac{1}{4}\sqrt{\frac{21}{4\pi}} \end{aligned}$$

Theseare the first few non vanishing moments. □

(b)
Solution:
The charge ensity is

$$\rho(\boldsymbol{x}) = \frac{q}{2\pi r^2}\left[\delta(r-a)\delta(1-\cos\theta) + \delta(r-a)\delta(1+\cos\theta) - \delta(r)\right]$$

The multipole moments are given by

$$\begin{aligned} q_{lm} &= \int r^l Y_l^m(\theta,\phi)\rho(\boldsymbol{x}) d^3x \\ &= qa^l P_l^m(0)\left[Y_l^m(0,0)^* + Y_l^m(\pi,0)^*\right] \end{aligned}$$

for $l > 0$ and $q_{00} = 0$. By azimuthal symmetry, only the $m = 0$ moments are non vanishing. Thus we get

$$\begin{aligned} q_{l_0} &= qa^l\sqrt{\frac{2l+1}{4\pi}}\left[P_l(1) + P_(-1)\right] \\ &= qa^l\left[1 + (-1)^l\right]\sqrt{\frac{2l+1}{4\pi}} \qquad l > 0 \end{aligned}$$

So, this leads to

$$\begin{aligned} q_{2,0} &= \sqrt{\frac{5}{\pi}}qa^2; \qquad q_{2,m\neq 0} = 0 \\ q_{4,0} &= \sqrt{\frac{9}{\pi}}qa^4; \qquad q_{4,m\neq 0} = 0 \end{aligned}$$

These are the moments. □

(c) For the charge distribution of the second set b write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x_0y plane as a function of distance from the origin for the distances grater than a.
Solution:
The expansion of the potential in terms of multipole coefficients is

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

Since we only have non-zero coefficients for $m = 0$ and l even we have

$$\begin{aligned} \Phi &= \frac{1}{4\pi\epsilon_0}\sum_{l=2,2,4}\frac{4\pi}{2l+1}q_{l0}\frac{Y_l^0(\theta,\phi)}{r^{l+1}} \\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=2,4\ldots}\frac{4\pi}{2l+1}qa^l\sqrt{\frac{2l+1}{\pi}}\sqrt{\frac{2l+1}{\pi}}\frac{P_l(\cos\theta)}{r^{l+1}} \\ &= \frac{q}{4\pi\epsilon_0}2\frac{a^l}{r^l+1}P_l(\cos\theta) \end{aligned}$$

The lowest order term is $l = 2$. And in the $x - y$ plane $\theta = \frac{\pi}{2}$ so we get

$$\Phi = -\frac{q}{4\pi\epsilon_0 a}\left(\frac{a}{r}\right)^3$$

This is the inverse cubic function whose graph is shown in Fig. (10.2) looks like. □

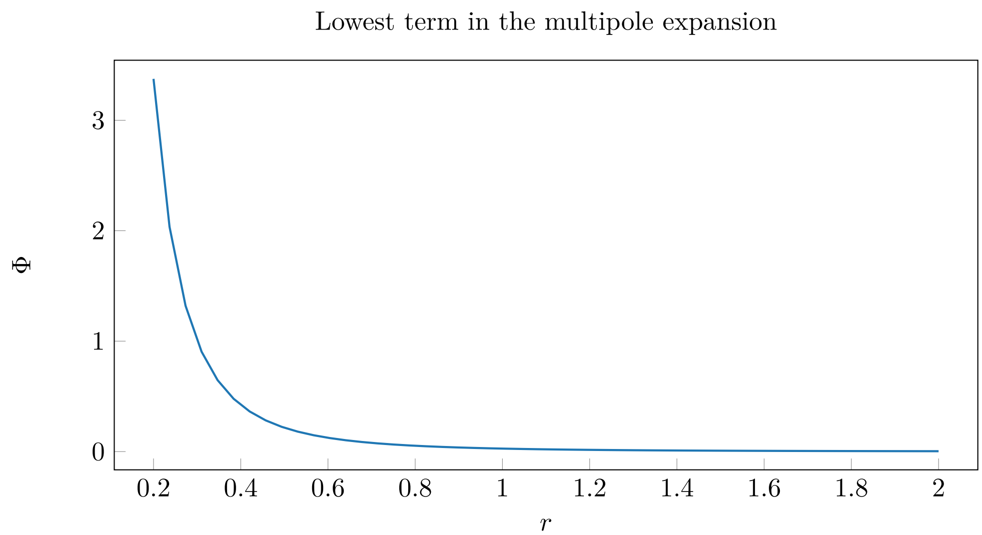


Figure 10.2: First term of multipole expansion.

(d) Calculate directly from Coloumb's law the exact potential for b in the $x-y$ plane. Plot it as a function of distance and compare withe the result found in part c.
Solution:
For the charges given we have in the cartesian coordinate system, in $x-y$ plane, if the distance from the origin to any point on the plane is r we get

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r} + \frac{1}{\sqrt{r^2+a^2}}\right)$$

Plotting this as a function r we get the plot in Fig. (10.3) □

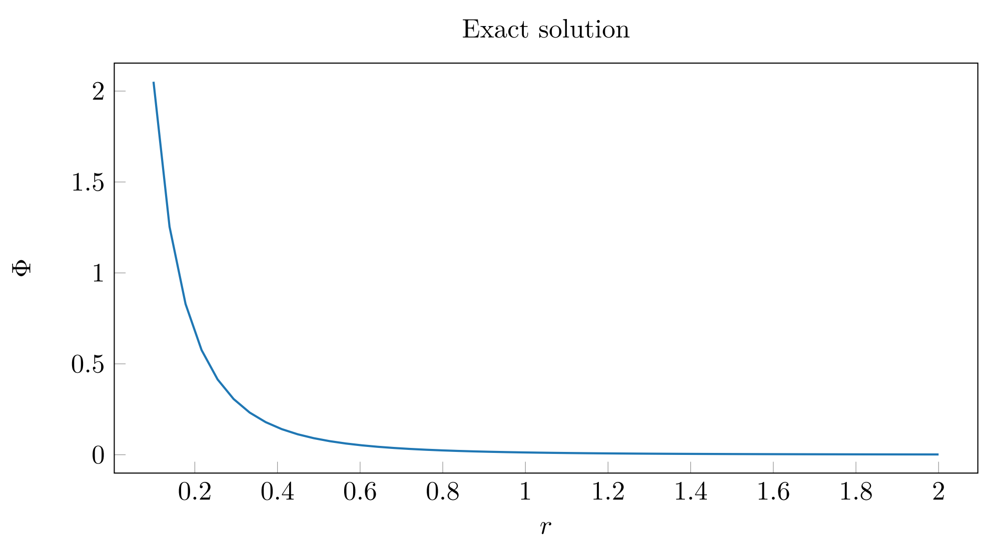


Figure 10.3: Exact solution

10.5.3. **(Jackson 4.9)** A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a ($a < d$) and dielectric constant ϵ/ϵ_0

(a) Find the potential at all points in space as an expansion in spherical harmonics.
Solution:
The charge at d inuces charge in the sphere. The induced charge produces the field inside the sphere. Again, using the general solution of Lapalace's equation in spherical system with azimuthal symetry we get

$$\Phi_{in} = \frac{q}{4\pi\epsilon}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{10.11}$$

We can chose the coordinate system such that the Z axis of our coordinate system passes through the charge and the center of sphere. WIh this. Outside the sphere the potential due to the chage is given by

$$\Phi_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty} B_l \frac{a^l}{r^{l+1}} P_l(\cos\theta) \tag{10.12}$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + B_l\left(\frac{a^l}{r^{l+1}}\right)\right] P_l(\cos\theta) \tag{10.13}$$

The component of electri field parallel to the surface of the sphere is

$$E_\theta^{\text{in}} = -\left.\frac{1}{r}\frac{\partial\Phi_{\text{in}}}{\partial\theta}\right|_{r=a} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[A_l\frac{r^l}{a^{l+1}}P_l'(\cos\theta)\sin\theta\right]_{r=a} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty} A_l\frac{1}{a}P_l'(\cos\theta)\sin\theta \tag{10.14}$$

Similarly the component outside the sphere is

$$E_\theta^{\text{out}} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + B_l\left(\frac{a^l}{r^{l+1}}\right)\right] P_l'(\cos\theta)\sin\theta\Bigg|_{r=a} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{a^l}{d^{l+1}} + \frac{B_l}{a}\right] P_l'(\cos\theta)\sin\theta \tag{10.15}$$

Equating (10.14) and (10.15) we get

$$\frac{q}{4\pi\epsilon}\frac{A_l}{a} = \frac{q}{4\pi\epsilon_0}\left[\frac{a^l}{d^{l+1}} + \frac{B_l}{a}\right] \implies A_l = \frac{\epsilon}{\epsilon_0}\left[\frac{a^{l+1}}{d^{l+1}} + B_l\right] \tag{10.16}$$

$$E_r^{\text{in}} = -\epsilon\left.\frac{\partial\Phi_{\text{in}}}{\partial r}\right|_{r=a} = \frac{q}{4\pi}\sum_{l=0}^{\infty}\left[A_l\frac{lr^{l-1}}{a^{l+1}}\right] P_l(\cos\theta)\Bigg|_{r=a} = \frac{q}{4\pi}\sum_{l=0}^{\infty}\left[A_l\frac{l}{a^2}\right] P_l(\cos\theta) \tag{10.17}$$

Similarly for the radial component of field outsie the sphere is

$$E_r^{\text{out}} = -\epsilon_0\left.\frac{\partial\Phi_{\text{out}}}{\partial r}\right|_{r=a} = \frac{q}{4\pi}\sum_{l=0}^{\infty}\left[\frac{la^{l-1}}{d^{l+1}} - B_l\frac{(l+1)a^{l+1}}{r^{l+2}}\right] P_l(\cos\theta)\Bigg|_{r=a} = \frac{q}{4\pi}\left[A_l\frac{la^{l-1}}{d^{l+1}} - B_l\frac{(l+1)}{a^2}\right] P_l(\cos\theta) \tag{10.18}$$

Equating (10.17) and (10.18) we get

$$\frac{q}{4\pi}\frac{A_l l}{a^2} = \frac{q}{4\pi}\left[\frac{la^{l-1}}{d^{l+1}} - B_l\frac{l+1}{a^2}\right] = A_l \implies \frac{a^{l+1}}{d^{l+1}} - B_l\frac{l+1}{l} \tag{10.19}$$

Solving two linear equations in A_l and B_l from (10.19) and (10.16) we get

$$B_l = \frac{\left(\frac{\epsilon_0}{\epsilon} - 1\right) l}{l + (l+1)\frac{\epsilon_0}{\epsilon}}\frac{a^{l+1}}{d^{l+1}} \tag{10.20}$$

$$A_l = \frac{2l+1}{l + (l+1)\frac{\epsilon_0}{\epsilon}}\frac{a^{l+1}}{d^{l+1}} \tag{10.21}$$

Substuting the coefficient in (10.20) an (10.12) we get

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon}\sum_{l=0}^{\infty}\frac{2l+1}{l+(l+1)\frac{\epsilon_0}{\epsilon}}\frac{r^l}{d^{l+1}}P_l(\cos\theta)$$

And similarly

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r^l}{d^{l+1}} + \frac{(\frac{\epsilon_0}{\epsilon}-1)l}{l+(1+l)\frac{\epsilon}{\epsilon_0}}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta)$$

These are the expression for the electri field inside and outside the sphere. □

(b) Calculate the rectangular components of the electric field near the center of the sphere.
Solution:
Inside the sphere, the first few terms are

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\frac{\epsilon_0}{\epsilon}}P_0(\cos\theta) + \frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\frac{r}{d}P_1(\cos\theta) + \frac{5}{2+3\frac{\epsilon_0}{\epsilon}}\frac{r^2}{d^2}P_2(\cos\theta) + \mathcal{O}(r^3)\right]$$

The radial radial component of the field is

$$\boldsymbol{E}_r = -\frac{\partial\Phi_{\text{in}}}{\partial r}\hat{\boldsymbol{r}} = -\frac{q}{4\pi\epsilon}\left[0 + \frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\frac{1}{d}P_1(\cos\theta) + \mathcal{O}(r)\right]\hat{\boldsymbol{r}} = -\frac{q}{4\pi\epsilon d}\left[\frac{3\cos\theta}{1+2\frac{\epsilon_0}{\epsilon}} + \mathcal{O}(r)\right]\hat{\boldsymbol{r}}$$

In the limit $r \to 0$ we get

$$\boldsymbol{E}_r = -\frac{q}{4\pi\epsilon d}\left[\frac{3\cos\theta}{1+e\frac{\epsilon_0}{\epsilon}}\right]\hat{\boldsymbol{r}}$$

Similarly the tangential (θ) component of field is

$$\boldsymbol{E}_\theta = -\frac{1}{r}\frac{\partial\Phi_{\text{in}}}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{-1}{r}\frac{q}{4\pi\epsilon}\left[0 + \frac{-3\sin\theta}{1+e\frac{\epsilon_0}{\epsilon}}\frac{r}{d} + \mathcal{O}(r)\right]\hat{\boldsymbol{\theta}} = \frac{q}{4\pi\epsilon d}\left[\frac{3\sin\theta}{1+2\frac{\epsilon_0}{\epsilon}} + \mathcal{O}(r)\right]\hat{\boldsymbol{\theta}}$$

In the limit $r \to 0$ we get

$$\boldsymbol{E}_\theta = \frac{q}{4\pi\epsilon d}\left[\frac{3\sin\theta}{1+2\frac{\epsilon_0}{\epsilon}}\right]\hat{\boldsymbol{\theta}}$$

Since the ϕ component of the field is 0 as the potential is independent of ϕ we get

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon d}\frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\left[-\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right] = \frac{q}{4\pi\epsilon d}\frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\hat{\boldsymbol{k}}$$

Where $\hat{\boldsymbol{k}}$ is the unit vector along $z-$axis. □

(c) Verify that, in this limit $\epsilon/\epsilon_0 \to \infty$, hour result is the same as that for conducting sphere
Solution:
In the limit $\epsilon/\epsilon_0 \to \infty$ we have

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0 d}$$

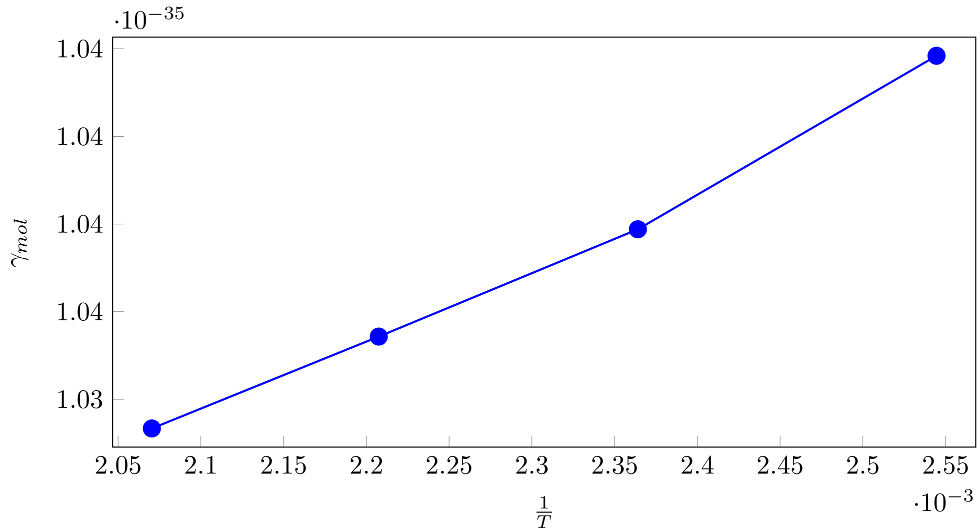
and

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0}\left[\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} - \sum_{l=1}^{\infty}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta)$$

We can invoke the spherical harmonics expansion in reerse and write the expression as

$$\frac{q}{4\pi\epsilon_0}\left[\frac{q/d}{r} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} - \frac{a}{|d\boldsymbol{r} - a^2\hat{\boldsymbol{r}}|}\right]$$

Whih is indeed the potential of a sphere outside the sphere □



10.6 Homework Six

10.6.1. **(Jackson 4.12)** Water vapor is a polar gas whose dielectric ocnstant exhibits and appreciable temperature dependence. The following table gives experimental data on this effect. Assuming that water vapor obeys the ideal gas law, calculate the molecular polarizability as a function of innverse temperature and plot it. From the solope of the curve, deduce a value for the permanent dipolemoment of the H_2O molecule.

T(K)	Pressure (cm Hg)	$(\frac{\epsilon}{\epsilon_0} - 1) \times 10^5$
393	56.49	400.2
423	60.93	371.7
453	65.34	348.8
483	69.75	328.7

Solution:
With the ideal gas equation we have

$$PV = NkT \qquad \Longrightarrow n = \frac{N}{V} = \frac{P}{kT}$$

By Clausis-Mossetti equation we have the molecular polarizability is given by

$$\gamma = \frac{3}{n}\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 - 2}\right) = \frac{3kT}{P}\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 - 2}\right)$$

Plotting this as a function of $\frac{1}{T}$ gives The slope is 8.9×10^{-35}

□

10.6.2. **(Jackson 4.13)** Two long, coaxial, cylindrical conductin surfaces of radii a and b are lowered vertically into a liquid dielectric. If the liquid raises an average height h between the electrodes when a potential difference V is established between them, show that th esusceptibility of the liquid is

$$\chi_e = \frac{b^2 - a^2 \rho g h \ln(b/a)}{\epsilon_0 V^2}$$

where ρ is the density of the liquid, g is the accleration due to gravity, and the susceptibility of th air is neglected.

Solution:
The total energy in the capacitor of capacitance C is given by

$$E = \frac{1}{2}CV^2$$

The capacitance of coaxial cylinder per unit length is given by

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

Let l is the length of the cylindrical conductors above the liquid, of which the liquid raises upto h. The section with $l - h$ is air filled and the section with height h above the liquid surface is dielectric filled. So the capacitance of each section gives

$$C_{air} = \frac{2\pi\epsilon_0(l-h)}{\ln(b/a)}; \qquad C_{\text{liquid}} = \frac{2\pi\epsilon h}{\ln(b/a)}$$

The total upward force on the raised liquid is thus

$$\begin{aligned} F = \frac{\mathrm{d}E}{\mathrm{d}h} = \frac{1}{2}V^2\frac{\mathrm{d}C}{\mathrm{d}h} &= \frac{1}{2}V^2\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{2\pi\epsilon_0(l-h)}{\ln(b/a)} + \frac{2\pi\epsilon h}{\ln(b/a)}\right) \\ &= \frac{\pi}{\ln(b/a)}\left[-\epsilon_0 + \epsilon\right] \end{aligned}$$

But we have $\epsilon = \epsilon_0 + \chi_e\epsilon_0$, so we get

$$F = \frac{\pi\chi_e\epsilon_0}{\ln(b/a)}$$

This force is balanced by the gravitational force in quilibrium which is given by

$$F = mg = \rho V_r g$$

The volume of raised liquied V_r is

$$V_r = \pi(b^2 - a^2)h$$

Thus

$$F = \rho\pi(b^2 - a^2)hg$$

Equating the forces

$$\begin{aligned} \frac{\pi\chi_e\epsilon_0}{\ln(b/a)} &= \rho\pi(b^2 - a^2)hg \\ \chi_e &= \frac{(b^2 - a^2)\rho gh\ln(b/a)}{\epsilon_0 V^2} \end{aligned}$$

This is the required expression. □

Chapter 11

Classical Electrodynamics II

11.1 Homework One

11.1.1. **(Jackson 6.1)** In three dimensions the solution to the wave equation (6.32) for a point source in space and time (a light flash at t' = 0, x' = 0) is a spherical shell disturbance of radius $R = ct$, namely the Green function $G^{(+)}$. It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake", even though the source is a "point" in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

(a) Starting with the retarded solution to the three-dimensional wave equation, show that the source $f(\boldsymbol{x}', t) = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave,

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function $[\Theta(\xi) = 0(1) \text{ if } \xi < (>)0]$

Solution:

The retarded solution is

$$\Psi(x, y, z, t) = \int \frac{[f(\boldsymbol{x}', t')]_{\text{ret}}}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' \tag{11.1}$$

Substuting the source function with the given delta functions we get

$$\Psi = \int \frac{\delta(x')\delta(y')\delta(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c})}{R} dx'dy'dz'$$

$$= \int_{-\infty}^{\infty} \frac{\frac{\delta(t-R}{c})}{R} dz'$$

Since we have cylindrical coordinate system we get

$$R = |\boldsymbol{x} - \boldsymbol{x}'| = \sqrt{\rho^2 + (z - z')^2} \text{ where } x' = y' = 0$$

This integral can be done with substitution. Supposing $u = z' + z$, we get $dz' = du$ and the limit stay the same

$$\Psi(\rho, t) = \int_{-\infty}^{\infty} \frac{\delta(t - \sqrt{\rho^2 + u^2}/c)}{\sqrt{\rho^2 + u^2}} du \tag{11.2}$$

Now this integral is of the form

$$\Psi(a) = \int \frac{\delta(f(x, a))}{g(x)} dx$$

making substitution of variable $f(x) = \beta$ we get $d\beta = f'(x)dx$ so that we get

$$\Psi(a) = \int \frac{\delta(\beta)}{g(x)} \frac{1}{f'(x)} d\beta$$

It is clear that the delta function only picks up values of x for which $\beta = f(x) = 0$. So the delta function reduces the integral to the sum of finite values for which $\beta = f(x) = 0$, let the solutions of $\beta = f(x) = 0$ be α_i, this makes,

$$\Psi(a) = \sum_i \frac{1}{g(\alpha_i) f'(\alpha_i)}$$

for this problem we have $f(u) = t - \frac{\sqrt{\rho^2 + u^2}}{c}$ whose zeros are

$$t - \frac{\sqrt{\rho^2 + \alpha_i^2}}{c} = 0 \qquad \implies \alpha_i = \pm\sqrt{c^2t^2 - \rho^2} \qquad \text{if } ct > \rho$$

there are no roots if $ct < \rho$ and the delta function is zero and the integral is identically zero. Also the derivative at the root is

$$f'(u) = \frac{u}{c\sqrt{\rho^2 + u^2}} \qquad \implies f'(\alpha_i) = \pm\frac{\sqrt{c^2t^2 - \rho^2}}{cct}$$

Substituting this in the integral (11.2), knowing that there are two values of α_i we get

$$\Psi(\rho, t) = \begin{cases} \frac{2c^2 t}{\sqrt{c^2t^2 - \rho^2}} \frac{1}{ct} & \text{if } ct \geq \rho \\ 0 & \text{if } ct \leq \rho \end{cases}$$

the two cases can be combined by using heaviside function

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}} = \frac{2c\Theta(ct - \sqrt{x^2 - y^2})}{\sqrt{c^2t^2 - x^2 - y^2}}$$

This is the required form of the wave

□

(b) Show that a "sheet" source, equivalent to a point pulse source at the origin in one space dimension produces a one dimensional wave proportional to

$$\Psi(x, t) = 2\pi c\Theta(ct - |x|)$$

Solution:
For the sheet source we expect a plane propagation of the wave. The source function for the sheet

source at some particular time $t' = 0$, let the $x' = 0$ plane be the source, so we can write the source function as

$$f(t', x') = \delta(x')\delta(t')$$

Using this source function to get the retarded time solution and substuting in (11.1) we get

$$\Psi(x, y, z, t) = \int_{-\infty}^{\infty} \frac{\delta(x')\delta(t')_{\text{ret}}}{R} dx'dy'dz'$$

Again we get $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$. Again similar to the previous rpoblem chaning of variables with $u = y - y', v = z - z'$ and recognizing that the delta function integral simply picks up $x' = 0$ we get

$$\Psi(x, y, z, t) = \int_{-\infty}^{\infty} \frac{\delta(t - \frac{\sqrt{x^2+u^2+v^2}}{c})}{\sqrt{x^2 + u^2 + v^2}} dudv$$

Since the integral has cylindrical symmetry when we have $\rho = \sqrt{u^2 + v^2}$ we can make cylindrical variable substitution to get

$$\Psi(\rho, \phi, z) = \int \frac{\delta t - \sqrt{\rho^2 + x^2}/c}{\sqrt{\rho^2 + x^2}} \rho d\rho d\phi$$

Due to cylindrical independence the phi integral is 2π and we are left with delta function integral similar to previous problem

$$\Psi(\boldsymbol{x}, t) = \int \frac{\delta(t - \sqrt{\rho^2 + x^2}/c)}{\sqrt{\rho^2 + x^2}} \rho d\rho$$

This again has a delta function inside the integral, and is non-zero only for the delta function equal to zero, the zeros of the expression inside the delta function, only give non zero values and the integral truns to a sum over these finite values of solution, the zeros of the delta are

$$t - \sqrt{\rho^2 + x^2}/c = 0 \qquad \Longrightarrow \qquad \rho = \pm\sqrt{c^2t^2 - x^2} \text{ if } ct > x$$

Also supposing $\beta = f(\rho) = t - \sqrt{\rho^2 + x^2}/c$ we get

$$d\beta = f'(\rho)d\rho \qquad d\beta = \frac{2\rho}{2c\sqrt{\rho^2 + x^2}} \qquad \Longrightarrow \rho d\rho = c\sqrt{\rho^2 + x^2}d\beta$$

Substuting these

$$\Psi(\boldsymbol{x}, t) = \int \frac{\delta(\beta)}{\sqrt{\rho^2 + x^2}} c\sqrt{\rho^2 + x^2}d\beta$$

Since there are two values of zeros of the funtion we have two terms in sum and we get

$$\Psi(\boldsymbol{x}, t) = c + c$$

By similar arguments as in the previous one we get non zero integral only if $ct > x$ we can write this using the Heaviside function

$$\Psi(\boldsymbol{x}, t) = 2c\Theta(ct - x)$$

This is the required function. □

11.1.2. **(Jackson 6.4)** A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi MR^3/3$ rotates about its magnetization axis with angular speed ω . IN the steady state no current flows in the conductor. The motion is non relativistic; the sphere has not excess charge on it.

(a) By considering Ohm's law in the moving conductor, show that the motion induces and electric field and a uniform volume charge density in the conductor $\rho = m\omega/\pi c^2 R^3$
Solution:
The magnetic moment of sphere is given by $\boldsymbol{m} = \boldsymbol{M}V$ where $V = \frac{2}{3}\pi R^3$ is the volume of sphere. Comparing it to the given magnetic moment we get that $\boldsymbol{M} = M\hat{\boldsymbol{z}}$. The magnetic flux density inside the sphere is given by

$$\boldsymbol{B} = \frac{2}{3}\mu_0 \boldsymbol{M} = \frac{\mu_0 m}{2\pi R^3}\hat{\boldsymbol{z}}$$

By ohm's law the current in the moving conductor is

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Since there is no current $J = 0$ which implies

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$$

Since the sphere has angular frequency $\boldsymbol{\omega}$, the translational velocity at r is given by $\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega} = \omega \boldsymbol{r} \times \hat{\boldsymbol{z}}$ thus we get

$$\boldsymbol{E} = \boldsymbol{r} \times \boldsymbol{\omega} \times \boldsymbol{B} = \frac{\mu_0 m}{2\pi R^3}\left[\hat{\boldsymbol{z}}(\hat{\boldsymbol{z}} \cdot \boldsymbol{r}) - \boldsymbol{r}(\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}})\right]$$

This simplifies to

$$\boldsymbol{E} = \frac{\mu_0 m\omega}{2\pi R^3}(\hat{\boldsymbol{z}}(\hat{\boldsymbol{z}} \cdot \boldsymbol{r}) - \boldsymbol{r})$$

This is the projection of vector $\boldsymbol{r}$ onto the horizontal axis, which in cylindrical system is

$$E_\rho = -\frac{\mu_0 m\omega\rho}{2\pi R^3}$$

Now that we have the field we can apply gauss' law to calculate the charge density

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

Sinc our field only has component along ρ we have

$$\rho = \epsilon_0 \frac{\partial E_\rho}{\partial \rho} = -\frac{\mu_0 \omega m \rho}{2\pi R^3}$$

This is the required volume charge density. □

(b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to shwo that the lowest possible electric multipolarity is quadrupole. Show that only quadrupole field exists outside that the quadrupule moment tensor has non vanishing components $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = \frac{-Q_{33}}{2}$.
Solution:
Since there is no charge inside the sphere the exterior can be described as the multipole expansion. Since there is no charge, the monopole moment which is the moment of total charge is zero. The electrostatic potential can be obtained as

$$\Phi(\rho) = -\int \boldsymbol{E}d\boldsymbol{l} = -\int E_\rho d\rho = \Phi_0 + \frac{\mu_0 m\omega\rho^2}{4\pi R^3}$$

This can be simplified by using the cartesian coordinate formulation as

$$\Phi(r,\theta) = \Phi_0 + \frac{\mu_0 m\omega}{4\pi R^3} r^2 \sin^2\theta.$$

Wrinting $\sin^2\theta$ in terms of legendre polynomials we get

$$\Phi(r,\theta) = \Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2 \left[P_0(\cos\theta) - p_2(\cos\theta)\right]$$

this simlifies to

$$\Phi(r,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2\right) P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R^3} r^2 P_2(\cos\theta)$$

At the surface of the sphere $r = R$ we get the potential as

$$\Phi(r,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2\right) P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R^3} r^2 P_2(\cos\theta)$$

Since the potential is azimuthally symmetric, we can write the external potential as a legendre polynomial series

$$V(\theta) = \sum_i A_l P_l(\cos\theta)$$

on the surface, and out side the surface the potential is

$$\Phi(,\theta) = \sum_l A_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

Since there is no charge the monopole term for $l = 0$ vanishes so we get

$$\Phi_0 = -\frac{\mu_0 m\omega}{6\pi R}$$

And the expression becomes.

$$\Phi(r,\theta) = -\frac{\mu_0 m\omega R^2}{6\pi r^3} P_2(\cos\theta)$$

Now tath we hae te exterior potential can be converted to expression with spherical harmonics

$$\Phi = -\sqrt{\frac{4\pi}{5}} \frac{\mu_0 m\omega R^2}{6\pi} \frac{Y_{20}(\theta,\phi)}{r^2}$$

The standard multipole expansion expression is

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{l=-\infty}^{\infty} \sum_{m=-l}^{l} \frac{2\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

compariosion gives

$$q_{20} = -4\pi\epsilon_0 \sqrt{\frac{5}{4\pi}} \frac{\mu_0 m\omega R^2}{6\pi} = -\sqrt{\frac{5}{4\pi}} \frac{2m\omega R^3}{3c^2}$$

The moment expression in cartesian coordinate system is given by

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}} q_{20} = -\frac{4m\omega R^2}{3c^2}, Q_{11} = Q_{22} = -\frac{1}{2} Q_{33}$$

this is teh required expression. □

(c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface charge density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2}\cdot\frac{4m\omega}{3c^2}\cdot\left[1-\frac{5}{2}P_2(\cos\theta)\right]$$

Solution:
the surface charge can be computed by using teh normal component as derivatives of potential. In the spherical coordinates we get

$$E_r^{\text{out}} = -\frac{\mu_0 m\omega R^2}{2\pi r^4}P_2(\cos\theta)$$
$$E_r^{\text{in}} = -\frac{\mu_0 m\omega r}{3\pi R^3}\left[p_o(\cos\theta) - P_2(\cos\theta)\right]$$

the surface charge is thus

$$\begin{aligned}\sigma = \epsilon_0\left(E_r^{\text{out}} - E_r^{\text{in}}\right)\Big|_{r=R} &= \frac{\mu_0\epsilon_0 m\omega}{3\pi R^2}\left[\frac{3}{2}P_2(\cos\theta) - (P_0(\cos\theta) - P_2(\cos\theta))\right]\\ &= \frac{m\omega}{3\pi c^2 R^3}\left[P_0(\cos\theta) - \frac{5}{2}P_2(\cos\theta)\right]\end{aligned}$$

This gives the required expression for teh surface charge density. □

(d) The rotating sphere serves aas a unipolar induction devie if a stationary circuit is attached by a slip ring to the pole and sliding contact to the equator. Show that hte line integral of the electric field from the equator contat to the pole contact is $\mathcal{E} = \mu_0 m\omega/4\pi R$
Solution:
The line integral is

$$\mathcal{E} = \int_{\text{equator}}^{\text{pol}} \boldsymbol{E}d\boldsymbol{l} = \Phi_{\text{equator}} - \Phi_{\text{pol}} = \Phi(\theta=\pi/2) - \Phi(\theta=0)$$

Substuting the value of theta in the expression for the potential we get

$$\mathcal{E} = -\frac{\mu_0 m\omega}{6\pi R}\left[P_2(0) - P_2(1)\right] = \frac{\mu_0 m\omega}{4\pi R}$$

This gives the required expression for the integral.

□

11.2 Homework Two

11.2.1. **(Jackson 6.11)** A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen

(a) From a law of conservation of linear momentum , show that the pressuer exerted on teh screen is equal to the field energy per unit volume in the wave.
Solution:
We can choose our coordinate system such that the z axis lies along the direction that the plane wave travels. Since electric amd magnetic fields are perpndicular to each other and to the direction of propagation the electric field and magnetic field become

$$\boldsymbol{E} = E\hat{\boldsymbol{\imath}} \qquad \boldsymbol{H} = H\hat{\boldsymbol{\jmath}}$$

The momentum conservation equation for j^{th} component of the momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{P}_{\text{fields}} + \boldsymbol{P}_{\text{mech}}\right)_j = \oint \sum_i T_{ij} n_i da \tag{11.3}$$

From the way we chose our coordinate system n only has component along the $\hat{\boldsymbol{k}}$ direction, the index for which is 3 so we can replace the summation by a single term

$$\sum_i T_{ij} n_i = T_{3j}$$

The stress energy tensor T_{ij} is given by

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} E^2 \delta_{ij}\right) + \mu_0 \left(H_i H_j - \frac{1}{2} H^2 \delta_{ij}\right)$$

Calculating the the component of the tensor in the rquired direciton we get

$$\begin{aligned} T_{j3} &= \epsilon_0 \left(E_3 E_j - \frac{1}{2} E^2 \delta_{3j}\right) + \mu_0 \left(H_3 H_j - \frac{1}{2} H^2 \delta_{3j}\right) \\ &= \frac{1}{2}\left(\epsilon_0 E^2 + \mu_0 H^2\right) \delta_{3j} \end{aligned}$$

Again by our choice of coordinate system the component of momentum is also along the z axis so the only non vanishing component of momentum is in that direction.

$$\begin{aligned} P_{j=3} &= \frac{1}{2}\left(\epsilon_0 E^2 + \mu_0 H^2\right) \delta_{3j}\bigg|_{j=3} \\ &= \frac{1}{2}\left(\epsilon_0 E^2 + \mu_0 H^2\right) \end{aligned}$$

The expression on the right is the expression for the energy density of electromagnetic wave so rthe expression can be written as

$$P_3 = u$$

where u is the energy density. Since the force is the change in momentum per unit time, and since the initial momentum is zero, we get

$$F = (P_3 - 0)/t = \bar{P}_3$$

where $\bar{}$ is the time averaged momentum. Which is equal to time averaged energy density, thus we get

$$F = \bar{u}$$

This shows that the energy density is energy density of the field. □

(b) In the neighborhood of the earth the flux of electromagnetic energy from the su is pproximately 1.4 kW/m^2. If an interplanetary "sailplane" had a sail of mass $\frac{1g}{m^2}$ of area and negligible other weight, what would be its maximum accleartion in meters persecond squared to the the solare readiation pressure? How does this compare with th eaccleartion due to solare "wind"(corpuscular radiation)?

Solution:

The flux relation to the energy density is $u = \frac{\text{flux}}{c}$ so we get

$$P = \frac{1.4 \times 10^3}{3.0 \times 10^8} = 5 \times 10^{-6} \frac{N}{m^2}$$

So the accleartion (a) can be calculated as

$$a = \frac{PA}{m} = \frac{P}{\frac{m}{A}} = \frac{5 \times 10^{-6}}{1 \times 10^{-3}} = 5 \times 10^{-3} \frac{m}{s^2}$$

The the accleartion is $a = 5 \times 10^{-}3 \frac{m}{s^2}$ □

11.2.2. **(Jackson 7.1)** For each set of Stokes parameters set $s_0 = 3, s_1 = -1, s_2 = 2, s_3 = -2$, deduce the amplitude of the electric field, up to an overall phase, in both linear polarization and circular polarization bases and make an accurate drawing similar to Fig. 7.4 showing the lengths of the axes of one of the ellipses and its orientation
Solution:
The stokes parameters are defined for linear polarization with the following relations

$$\begin{aligned}
s_0 &= |E_1|^2 + |E_2|^2 \\
s_1 &= |E_1|^2 - |E_2|^2 \\
s_2 &= 2\text{Re}(E_1^* E_2) = 2|E_1||E_2| \cos(\theta_2 - \theta_1) \\
s_3 &= 2\text{Im}(E_1^* E_2) = 2|E_1||E_2| \sin(\theta_2 - \theta_1)
\end{aligned}$$

Inverting these relations we get

$$|E_1| = \sqrt{\frac{s_0 + s_1}{2}} = 2 \qquad |E_2| = \sqrt{\frac{s_0 - s_1}{2}} = \sqrt{2}$$
$$\theta_2 - \theta_1 = \text{acos}\left(\frac{s_2}{\sqrt{s_0^2 - s_1^2}}\right) = \frac{\pi}{4}$$

With these paameters we get the components of electric field as

$$\boldsymbol{E} = \left(|E_1|e^{i\theta_1}, |E_2|e^{i\theta_2}\right) = e^{i\theta_1}\left(|E_1|, |E_2|e^{i\theta_2 - i\theta_1}\right)$$

Sice the phase factor in fron is arbitrary we can ignore it because we can always achieve zero phase factor by rotation of choice of axes. Similarly for the circular polarization case we have the stokes parameters defined as

$$\begin{aligned}
s_0 &= |E_+|^2 + |E_-|^2 \\
s_1 &= 2|E_+||E_-| \cos(\theta_- - \theta_+) \\
s_2 &= 2|E_+||E_-| \sin(\theta_- - \theta_+)
\end{aligned}$$

Similarly inverting these field amplitudes in terms of parameters give

$$|E_+| = \sqrt{\frac{s_0 + s_3}{2}} = \frac{1}{\sqrt{2}} \qquad |E_-| = \sqrt{\frac{s_0 - s_3}{2}} = \sqrt{\frac{5}{2}}$$
$$\theta_- - \theta_+ = \text{acos}\left(\frac{s_1}{\sqrt{s_0^2 - s_3^2}}\right) = \text{acos}\left(\frac{-3}{\sqrt{5}}\right)$$

Now the field cmponents are

$$\boldsymbol{E} = \left(|E_1|e^{i\theta_1}, |E_2|e^{i\theta_2}\right) = e^{i\theta_1}\left(|E_1|, |E_2|e^{i\theta_2 - i\theta_1}\right)$$

With the parameter for E_1 and E_2 and the phase difference the diagram can be plotted. □

11.2.3. **(Jackson 7.3)** Two plane semi-infinite slabs of the same uniform , isotropic, nonpermeable, lossless dielectric with index of refraction n are parallel and separated by an air gap ($n = 1$) with width d . A plane electromagnetic wave of frequency ω is incident on the gap from one of teh slabs with the angle fo incidence i . For linear polarization *both* parallel and perpendicular to the plane of incidence

(a) Calculate the ratio of power transmitted into the second slab to the inciden power and the ratio of reflected to incident power.
Solution:
Let i is the incident angle and r is the angle of refraction by snells law we have

$$n \sin i = \sin r$$

where n is the refractive index. We can rearrange thi sto get

$$\cos r = \sqrt{1 - \sin^2 r} = \sqrt{1 - n^2 \sin^2 i}$$

The value of $\cos r$ is purely imaginary when i is greater than critical angle for total internal reflection. To find the transmittedn and replected components in terms of th eincident component we can use the interface matching. In the first interface

$$\begin{aligned} E_p &= E_i + E_r = E_+ + E_- \\ H_p &= n(E_i - E_r)\cos i = (E_+ - E_-)\cos r \end{aligned}$$

Here we have E_p and H_p are the parallel components of electric and magnetic field.
In the second interface we have

$$\begin{aligned} E_+ e^{i\boldsymbol{k}\cdot\boldsymbol{d}} + E_- e^{-i\boldsymbol{k}\cdot\boldsymbol{d}} &= E_t \\ (E_+ e^{i\boldsymbol{k}\cdot\boldsymbol{d}} - E_- e^{i\boldsymbol{k}\cdot\boldsymbol{d}})\cos r &= nE_t \cos i \end{aligned}$$

Solving for E_+ and E_- in terms of E_r and E_i we get

$$\begin{aligned} E_+ &= \frac{1}{2}E_i\left(1 + \frac{n\cos i}{\cos r}\right) + \frac{1}{2}E_r\left(1 - \frac{n\cos i}{\cos r}\right) \\ E_- &= \frac{1}{3}E_i\left(1 - \frac{n\cos i}{\cos r}\right) + \frac{1}{2}E_r\left(1 + \frac{n\cos i}{\cos r}\right) \end{aligned} \tag{11.4}$$

Similarly the condition with the second interface can be solved to get

$$\begin{aligned} E_+ &= \frac{1}{2}e^{i\boldsymbol{k}\cdot\boldsymbol{d}}E_t\left(1 + \frac{n\cos i}{\cos r}\right) \\ E_- &= \frac{1}{2}e^{i\boldsymbol{k}\cdot\boldsymbol{d}}E_t\left(1 - \frac{n\cos i}{\cos r}\right) \end{aligned} \tag{11.5}$$

Let us write $\epsilon = \frac{n\cos i}{\cos r}$ Euation. (11.4) and (11.5) can be solved to get

$$\frac{E_t}{E_i} = \frac{4\epsilon}{(1+\epsilon)^2 e^{-i\boldsymbol{k}\cdot\boldsymbol{d}} - (1-\epsilon)^2 e^{i\boldsymbol{k}\cdot\boldsymbol{d}}} \tag{11.6}$$

$$\frac{E_r}{E_i} = \frac{(1-\epsilon^2)(e^{i\boldsymbol{d}\cdot\boldsymbol{k}})}{(1+\epsilon)^2 e^{i\boldsymbol{k}\cdot\boldsymbol{d}} - (1-\epsilon)^2 e^{i\boldsymbol{k}\cdot\boldsymbol{d}}} \tag{11.7}$$

This gives the ratio of transmitted to reflected amplitudes. The power is proportional to the square amplitudes so the ratio of transmitted power to the incident power is

$$\frac{P_t}{P_i} = \frac{E_t^2}{E_i^2} = \left[\frac{E_t}{E_i}\right]^2$$

and similarly the reflected power ratio is

$$\frac{P_r}{P_i} = \frac{E_r^2}{E_i^2} = \left[\frac{E_r}{E_i}\right]^2$$

These are the required ratios where the ratios of amplitudes are calculated. □

(b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d in units of wavelength in the gap.
Solution:
In the equations (11.6) and (11.7) we can write the ration ϵ and the phase $\boldsymbol{k}\cdot\boldsymbol{d}$ as purely imaginary numbers and simplify those quations to get the function of th ratios. So assuming the phase and the ratio to be complex we get

$$\epsilon = i\alpha \qquad \boldsymbol{k}\cdot\boldsymbol{d} = i\beta$$

Using these in (11.6) and (11.7) we get

$$\frac{T_t}{T_i} = \left[\frac{2i\alpha}{2i\alpha\cos h\beta + (1-\alpha^2)\sinh\beta}\right]^2 = \frac{4\alpha^2}{4\alpha^2 + (1+\alpha^2)\sinh^2\beta}$$

and smimilarly the ratio of reflected to transmitted power is

$$\frac{T_r}{T_i} = \frac{(1+\alpha^2)^2\sinh^2\beta}{4\alpha^2 + (1+\alpha^2)\sinh^2\beta}$$

Substuting $\beta = kd$ and also $n = 1$ we get

$$\frac{T_r}{T_i} = \frac{(1+\alpha^2)^2\sinh^2 kd}{4\alpha^2 + (1+\alpha^2)\sinh^2 kd}$$

Graphing this function as a function of $\frac{d}{\lambda}$ we get.

□

11.3 Homework Three

11.3.1. **(Jackson 7.12)** The time dependence of electrical distrubaances in good conductors is governed by the frequency-dependent conductivity. Consider longitudinal electric fields in a conductor, using Ohm's law, the contunuity equation, and the different form of Coulomb's law.

(a) Show that the time-Fourier transformed charge density satisfies the equation

$$[\sigma(\omega) - i\omega\epsilon_0]\,\rho(\boldsymbol{x},\omega) = 0$$

Solution:
Let us assume the time varying quantities be charge density $\rho(t)$, current density $J(t)$ and electric field $E(t)$. Taking the fourier transform to take to frequency space

$$\rho(\omega) = \frac{1}{\sqrt{2\pi}}\int \rho(t)e^{i\omega t}dt$$
$$\boldsymbol{J}(\omega) = \frac{1}{\sqrt{2\pi}}\int \boldsymbol{J}(t)e^{i\omega t}dt$$
$$\boldsymbol{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int \boldsymbol{E}(t)e^{i\omega t}dt$$

Now The contuinity equation

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$$

In the frequency space, this becomes

$$\nabla \cdot \boldsymbol{J}(\omega) = i\omega\rho(\omega)$$

The Ohm's law relates chage current density and electric field as,

$$\boldsymbol{J}(\omega) = \sigma(\omega)\boldsymbol{E}(\omega)$$

The coloumbs law can be used express the relation between the electric field and charge density as

$$\nabla \cdot \boldsymbol{E}(\omega) = \frac{\rho(\omega)}{\epsilon_0}$$

Combining all these we obtain

$$(\sigma(\omega) - i\omega\epsilon_0)\,\rho(\omega) = 0$$

This is the required expression. □

(b) Using the representation $\sigma(\omega) = \sigma_0/(1 - i\omega\tau)$ where $\sigma_0 = \epsilon_0\omega_p^2\tau$ and τ is a dampinng time, show that the approximation $\omega_p\tau \gg 1$ any initial distrubance will oscillate with plasma frqueynyc and decay amplitude with a decay constatn $\lambda = 1/2\tau$.
Solution:
Using the representation $\sigma(\omega) = \sigma_0(1 - i\omega\tau)$ we get

$$\left(\frac{\sigma_0}{1 - i\omega\tau} - i\omega\epsilon_0\right)\rho(\omega) = 0$$

substuting $\sigma_0 = \epsilon_0\omega_p^2\tau$

$$\left[\frac{\omega_p^2\tau}{1 - i\omega\tau} - i\omega\right] = 0$$

this is a quadratic equation in ω which can be rearranged to get $\tau\omega^2 + i\omega - \omega_p^2\tau = 0$. The solutions are

$$\omega = \frac{-i \pm \sqrt{4\tau^2\omega_p^2 - 1}}{2\tau}$$

Using the given approximation $\omega_p\tau \gg 1$ we obtain

$$\omega = \pm\omega_p - \frac{i}{2\tau}$$

This shows that in frequency space the signal is delayed by $\frac{1}{2\tau}$. Reverting back to time space with inverse fouerir transform we get

$$f(t) = \mathcal{F}^{-1}F(\omega_p - i\frac{1}{2\tau})$$
$$f(t) = f_0(t)e^{-t/2\tau}$$

This shows that the signal is decays at the rate $\frac{1}{2\tau}$ □

11.3.2. **(Jackson 7.19)** An approximately monochormatic plane wave packet in one dimension has the instantaneous form $u(x,0) = f(x)e^{k_0 x}$, with $f(x)$ the modulation envelope. For each of teh forms $f(x)$ below, calculate the wave number spectrum $|A(k)|^2$ of the packet, sketch $|u(x,0)|^2$ and $|A(k)|^2$, evaluate explicitly the rms deviations from the means Δx and Δk

(a) $f(x) = Ne^{\alpha|x|/2}$
Solution:
The initial waveform for this problem is $u(x,0) = Ne^{k_0 x = \alpha|x|/2}$ The wave number spectrum can be obtained as

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)e^{ikx} ds$$
$$= \frac{1}{\sqrt{2\pi}} N \int_{-\infty}^{\infty} e^{ikx + ik_0 x - \alpha\frac{|x|}{2}} dx$$

This integral is a function of α and since it is even function of x we can write above integral as

$$A(k) = \frac{1}{\sqrt{2\pi}} 2N \left[\int_0^{\infty} \cos(k-k_0)xe^{-\alpha x/2} \right]$$

This integral can be computed and the final expression fo rthe integral gives

$$A(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{N\alpha}{\alpha^2/4 + (k-k_0)^2} \right]$$

The mean square value for a function $f(x)$ is given by the expression

$$\text{Mean Square } = \frac{\int_{-\infty}^{\infty} x^2 \left[f(x)\right]^2 dx}{\int_{-\infty}^{\infty} \left[f(x)\right]^2 dx}$$

For the mean squred deviation of x we can write

$$\sigma_x^2 = \frac{\int_{-\infty}^{\infty} x^2 e^{-\alpha|x|} dx}{\int_{-\infty}^{\infty} e^{\alpha|x|} dx}$$

These integrals can be calulated with gamma functions, and the final result after integration is

$$\Delta x = \frac{\sqrt{2}}{\alpha}$$

Similarly with same token for the spread of $A(x)$ we obtain

$$\Delta k = \sqrt{\frac{\int_{-\infty}^{\infty} k^2 \left[\frac{1}{\alpha^2/4 + k^2}\right]^2 dk}{\int_{-\infty}^{\infty} \left[\frac{1}{\alpha^2/4 + k^2}\right]^2 dk}}$$

This integral was obtained using computer algebra system and the final expression is

$$\Delta k = \frac{\alpha}{2}$$

Checking for the product of $\Delta x \Delta k$ we get

$$\Delta x \Delta k = \sqrt{2}/2 = \frac{1}{\sqrt{2}} \geq \frac{1}{2}$$

□

(b) $f(x) = Ne^{\alpha^2 x^2}/4$
Solution:
Taking the fouerie transform to get the frequency component functions

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{ikx} dx$$

This can be integrated for the geven initial shape as

$$A(k) = \frac{1}{\sqrt{2\pi}} N \int_{-\infty}^{\infty} e^{ik_0 x - ikx - \alpha^2 x^2/4}$$

Thsi can be calculate to obtain

$$A(k) = N\sqrt{2/\alpha^2} e^{-(k-K_0)^2/\alpha^2}$$

The spread can bew be similarly calculaged as above

$$\Delta x = \sqrt{\frac{\int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2/2} dx}{\int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} dx}}$$

The integrals can be calculated using gamm function identities and the final expression (with computer algebra system used) is

$$\Delta x = \frac{1}{\alpha}$$

Similarly the spread in the frequency component can be calculated

$$\Delta k = \sqrt{\frac{\int_{-\infty}^{\infty} k^2 e^{-2k^2/\alpha^2} dk}{\int_{-\infty}^{\infty} e^{-2k^2/\alpha^2} dk}}$$

This was also solved using computer algebra system to obtain

$$\Delta k = \frac{\alpha}{2}$$

For this signal also the inequality $\Delta x \Delta k \geq \frac{1}{2}$ holds as

$$\Delta x \Delta k = \frac{1}{\alpha} \cdot \frac{\alpha}{2} = \frac{1}{2} \geq \frac{1}{2}$$

So both the wave train satisfy the uncertainity principle. □

11.3.3. **(Jackson 8.2)** A transmission line sonsisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric (μ, ϵ). A TEM mode is propagated along this line,

(a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\epsilon}} \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

where H_0 is the peak value of the azimuthal magnetic field and the surface of the innser conductor.
Solution:
By definition a TEM mode is a signle-frequency wave cmponent with both the electric field and magnetic field transverse to the directio of propagation along the waveaxis. The innder conductor has to have some charge per unit length, say λ. With a cylindrical gaussian surface around the inner conductor we find the electric field is

$$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon\rho}\hat{\boldsymbol{\rho}}$$

Since the waveguide axis is along the z axis, the magnetic field can be obtained from electric field as

$$\boldsymbol{B} = \sqrt{\mu\epsilon}\hat{\boldsymbol{z}} \times \boldsymbol{E} = \sqrt{\mu\epsilon}\frac{\lambda}{2\pi\epsilon\rho}\hat{\boldsymbol{\phi}}$$

Since given in the problem that H_0 is the peak value of magnetic field in the inner conductor, we obtain H_0 as

$$H_0 = \boldsymbol{H}(\rho = a) = \boldsymbol{B}(\rho = a)\frac{1}{\mu\epsilon} = \frac{1}{\sqrt{\mu\epsilon}}\frac{\lambda}{2\pi a}$$

This expression gives the total charge per unit length equal to

$$\lambda = 2\pi a H_0 \sqrt{\mu\epsilon}$$

Substuting this in the expression for electric field we get

$$\boldsymbol{E} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{q}{\rho}\hat{\boldsymbol{\rho}} \qquad \boldsymbol{B} = \mu H_0 \frac{a}{\rho}\hat{\boldsymbol{\phi}}$$

These two fields are correct for the static problem. Introducing the time dependence in the waveguide we obtain

$$\boldsymbol{E} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{q}{\rho} e^{ikz - i\omega t}\hat{\boldsymbol{\rho}} \qquad \boldsymbol{B} = \mu H_0 \frac{a}{\rho} e^{ikz - i\omega t}\hat{\boldsymbol{\phi}}$$

Now we can calculate the energy flux using the poynting vector as

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$
$$\boldsymbol{S} = \frac{1}{\mu}\left[\sqrt{\mu/\epsilon} H_0 \frac{a}{\rho} e^{ikz - i\omega t}\right] \times \left[\mu H_0 \frac{a}{\rho} e^{ikz - i\omega t}\right]$$

Since in general we the quantity H_0 is a complex number we can write this as

$$H_0 = |H_0|e^{i\theta}$$

Substuting this in aboe expression and carrying out the corss product we get

$$\boldsymbol{S} = \sqrt{\frac{\mu}{\epsilon}}|H_0|^2\frac{a^2}{\rho^2}\cos^2(kz - \omega t + \theta)\,\hat{\boldsymbol{z}}$$

The time averaged power flux is thus the average of above expression. But the average of $\cos^2$ is

$$\langle\cos^2\alpha\rangle = \frac{1}{2}$$

So we get

$$\langle\boldsymbol{S}\rangle = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2\frac{a^2}{z^2}\hat{\boldsymbol{z}}$$

The total power can now be obtained by integrating the power flux over the whole area

$$\begin{aligned} P &= \oint_S \hat{\boldsymbol{z}}\cdot\langle\boldsymbol{S}\rangle\,dA \\ &= \int_0^{2\pi}\int_a^b \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2\frac{a^2}{\rho^2}\rho d\rho d\phi \end{aligned}$$

The integral in ϕ is just the value 2π and the rho integral is just logarithm. So we get

$$P = \sqrt{\frac{\mu}{\epsilon}}\pi a^2|H_0|^2\ln\left(\frac{b}{a}\right) \tag{11.8}$$

This is the requried power flow. □

(b) Show that hte transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\frac{1}{b}+\frac{1}{b}}{\ln\left(\frac{b}{a}\right)}$$

Solution:
The rate of power loss per unit area with skin depth δ is given by

$$\frac{\mathrm{d}P}{\mathrm{d}a} = \frac{1}{4}\mu_c\omega\delta\left|\boldsymbol{H}_{\parallel}\right|^2$$

The area element in can be written as

$$da = \rho d\phi dz$$

Using this expression in the power flow equation we get

$$\frac{\mathrm{d}P}{\mathrm{d}a}\rho d\phi = \frac{1}{4}\mu_c\omega\delta\int_0^{2\pi}\left|\boldsymbol{H}_{\parallel}\right|^2\rho d\phi$$

There are two boundaries the surface so we get

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{\pi}{2}\mu_c\omega\delta\left[\left|\boldsymbol{H}_{\parallel}(a)\right|^2 a + b\left|\boldsymbol{H}_{\parallel}(b)\right|^2\right]$$

The magnetic field part of the expression can be substutited to get

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{\pi}{\sigma\delta}|H_0|^2 a\left[1+\frac{a}{b}\right] \tag{11.9}$$

As given in the problem , assuming the power loss along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

Differentiating with respect to z we get

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} = -2\gamma P \qquad \Longrightarrow \gamma = \frac{-1}{2P}\frac{\mathrm{d}P}{\mathrm{d}z}$$

Substuting P from (11.8) and its derivative from (11.9) we get

$$\gamma = \frac{1}{\sqrt{\frac{\mu}{\epsilon}}\pi a^2|H_0|^2\ln\left(\frac{b}{a}\right)}\left(\frac{\pi}{\sigma\delta}|H_0|^2 a\left[1+\frac{a}{b}\right]\right)$$

Simplification yields

$$\gamma = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

This is the required expression. □

11.4 Homework Four

11.4.1. **(Jackson 8.4)** Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ

(a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in term of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part assume that the conductivity of the cylinder is infinite.

Solution:

The eivenvalue equation for both the TE and TM mode is

$$\left(\nabla_t^2 + \gamma^2\right)\psi(r,\phi) = 0$$

where $\psi(R,\phi) = 0$. For TE mode there is no axial electric field, so we can solve for B_z. There are no charges and currents in the waveguide so they obety the homogenous wave equation

$$\nabla^2 B_z - \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2} = 0$$

The wave is free along the axis of waveguide. so we can ssume that the solution for the magnetic field has harmonic dependence in time in the direction of propogation thus we can write

$$B = B_z e^{ikz - i\omega t}$$

Substuting this in the expression for the homognneous equation we get

$$\nabla_t^2 B_z = \left(\frac{\omega^2}{c^2} - k^2\right) B_z$$

The laplacian operator in this expression is only in the transerse directionn. Because of the cylindrical symmetri we can write the lapalacian in the cylindrical coordinate system as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 B_z}{\partial \phi^2} = \left(k^2 - \frac{\omega^2}{c^2}\right) B_z$$

Making a substution

$$k' = \left(\frac{\omega^2}{c^2} - k^2\right)$$

If we assume the solution of the magnetic field $B_z = R(k'r)e^{im\phi}$ we get

$$r^2\frac{\partial^2 R(k'r)}{\partial r^2} + r\frac{\partial R(k'r)}{\partial r} + \left(k^2 r^2 - m^2\right) R(kr) = 0$$

This differential equation can be converted to a bessel differential equation with $kr = x$. The equation then becomes

$$x^2\frac{\partial^2 R(x)}{\partial x^2} + x\frac{\partial R(x)}{\partial x} + \left(x^2 - ^2\right) R(x) = 0$$

This is besseldifferential equatio. the solution of this equation is

$$R(x) = AJ_m(x) + BN_m(x)$$

Substuting this in the magnetic field expression we get

$$B_z = \left(A_m(x) + BN_m(x)\right) e^{ikz-\omega t+m\phi}$$

Since the functions $N_m(x)$ blow up at $x = 0$, and that the field is finite at the axis we have to have $B = 0$. The solution then becomes

$$B_z = AJ_m(k'r)e^{i(kz-\omega t+m\phi)}$$

At the surface of the perfect conductior constuting the wallsof eh waveguide, we have theboundary condidtion

$$\left.\frac{\partial B_z}{\partial r}\right|_{r=R} = 1$$

Applying this condition we get

$$\left(\frac{\partial}{\partial r}\left(J_m(k'r)\right)\right)_{r=R} = 0$$

The zeros of the equation are simply the zeros of derivatives of bessel functions. Assuming the zeros are α_{mn} we get

$$k'r = \alpha_{mn} \implies k' = \frac{\alpha_{mn}}{R}$$

Substuting this for the expression relating k and k' we get

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\alpha_{mn}^2}{R^2}}$$

Thus the magnetic field becomes

$$B_z = AJ_m(\alpha_{mn}\frac{r}{R})e^{i(kz-\omega t+m\phi)}$$

For TE mode the axial electric field obeys same equation and we get similar differential equation whose solution is

$$E_z = AJ_m(k'r)e^{i(kz-\omega t+m\phi)}$$

The boundary condition is tha the electric field is zero at the walls $E_z(r=R)=0$ so we get Now instead of the zeros of derivatives of bessel function the zeros are at the zeros of bessel function β_{mn} so we get

$$k' = \frac{\beta_{mn}}{R}$$

The solution then becomes

$$E_z = AJ_m(\beta_{mn}\frac{r}{R})e^{i(kz-\omega t+m\phi)}$$

The cutoff frequencies are the frequencies where the wavenumber equals zero. So we get

$$\omega_{mn} = c\frac{\beta_{mn}}{R}\text{for TE mode}$$
$$\omega_{mn} = c\frac{\alpha_{mn}}{R}\text{for TM mode}$$

These are the required cutoff frequencies for TE mode and TM mode. □

(b) Calculate for TM mode the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.
Solution:
For TM mode, the power loss is given by

$$-\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \oint_c \frac{1}{\mu^2\omega_{mn}^2}\left|\frac{\partial\psi}{\partial a}\right| dl$$

□

11.4.2. **(Jackson 8.6)** A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with the flat end faces. Determine the resonant frequencies of the cavity for all types of waves. With $\frac{1}{\sqrt{\mu\epsilon}R}$ as a unit of frequency. Plot the lowest resonant frequencies of each type as s function of $\frac{R}{L}$ for $0 < \frac{R}{L} < 2$. Does the same mode have the lowest frequency for $\frac{R}{L}$?
Solution:
For the cavity, the normal modes in TM modes are given by

$$\psi(,\phi) = E_0 J_m(\gamma_{mn} r)e^{\pm im\phi} \qquad \text{where}\gamma_{mn} = \frac{x_{mn}}{R}$$

Here x_{mn} are the zeros of bessle function J_m. As given in Jackson eq. 8.81 we get the resonant frequency

$$w_{mnp} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{x_{mn}^2 + \left(\frac{p\pi R}{L}\right)^2}$$

The zeros of bessel are

$$x_{01} = 2.405, x_{12} = 3.832, x_{21} = 5.136 and$$

□

11.5 Homework Five

11.5.1. **(Jackson 9.3)** Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulation gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long wavelength limit, find the radiation fields, the angular distribution of radiated power and the total radiated power from the sphere.

Solution:

Two opposite charged halves of sphere creates a dipole so the dipole term in the potential expansion is the dominant term. The dominant term on the potential expansion in terms of Legendre polynomial expansion is

$$\Phi = V\frac{3}{2}\frac{R^2}{r^2}\cos\theta$$

The potential due to electric dipole pointing iz the positive z direction is given by

$$\Phi_{\text{dip}}\frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta$$

The donimant term must be equal to the dipole potential. Equating these

$$V\frac{3}{2}\frac{R^2}{r^2}\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta$$
$$\implies \boldsymbol{p} = 6\pi\epsilon_0 VR^2\hat{\boldsymbol{z}}$$

The potential in the sphere is oscillation with the frequency ω as $\cos\omega t$, The magnetic field of such oscillating field can be written as

$$\boldsymbol{B} = \frac{\mu_0 ck^2p}{4\pi}\left(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{p}}\right)\frac{e^{i(kr-\omega t)}}{r}$$

Subsisting the value of the dipole moment we get

$$\boldsymbol{B} = -\frac{3}{2}\frac{Vk^2R^2}{c}\frac{e^{i(kr-\omega t)}}{r}\sin\theta\hat{\boldsymbol{\phi}}$$

The electric field is similarly given by

$$\boldsymbol{E} = -\frac{k^2p}{4\pi\epsilon_0}\hat{\boldsymbol{k}}\times\left(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{p}}\right)\frac{e^{i(kr-\omega t)}}{r}$$

Simplifying the vector cross products we simplify this down to

$$\boldsymbol{E} = -\frac{3}{2}Vk^2R^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\hat{\boldsymbol{\theta}}$$

Now the overall radiated power per solid angle is given by

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{2}\mathrm{Re}\left(r^2\hat{\boldsymbol{r}}\cdot\boldsymbol{E}\times\boldsymbol{H}^*\right)$$

Subsisting the values of electric field and magnetic field in this expression we get

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{2}\mathrm{Re}\left(r^2\hat{\boldsymbol{r}}\cdot\left\{-\frac{3}{2}Vk^2R^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\hat{\boldsymbol{\theta}}\right\}\times\left\{-\frac{1}{\mu_0}\frac{3}{2}\frac{Vk^2R^2}{2}\frac{e^{i(kr-\omega t)}}{r}\sin\theta\hat{\boldsymbol{\phi}}\right\}\right)$$

Since $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}}$, the above expression simplifies to

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{9}{8}\frac{V^2k^4R^4}{\mu_0 c}\sin^2\theta$$

The total radiated power is thus the integral of the above expression over the total solid angle in the entire spherical shell

$$P = \oint \frac{\mathrm{d}P}{\mathrm{d}\Omega} d\Omega = \oint \frac{9}{8} \frac{V^2 k^4 R^4}{\mu_0 c} \sin^2 \theta d\Omega$$

Integral of of the quantity $\sin^2 \theta$ over the total solid angle is just $\frac{8}{3}$ thus giving us the final expression

$$P = \frac{3\pi V^2 K^4 R^4}{\mu_0 c}$$

This gives the total radiated power. □

Chapter 12

General Relativity

12.1 Homework One

12.1.1. **(Geometrized Units)** Express each of the pollowing quantities in two ways: i) in m^n, as meters raised to some appropriate power, and ii) in kg^n as kilograms raised to the appropriate power.

(a) The momentum of an electron moving at $0.8c$.
Solution:
The gamma factor γ is

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - .8^2}} = 1.67$$

The mass of electron is $m_e = 1.21 \times 10^{-31}$kg. So th momentum is

$$p = mv\gamma = 9.1 \times 10^{-31} \cdot 0.8 \cdot 1.67 = 1.21 \times 10^{-30} kg$$

Since the conversion factor is $1m = 1.35 \times 10^{27} kg$ we get

$$p = 1.21 \times 10^{-30} \left(1.35 \times 10^{27}\right)^{-1} = 8.96 \times 10^{-58} m$$

These are the required values of momentum in each unit. □

(b) The age of universe (13.8)Gy
Solution:
The age(A) in seconds is

$$A = 13.8 \times 10^9 \cdot 365 \cdot 24 \cdot 60 \cdot 60 = 4.35 \times 10^{17} s$$

The conversion factor is $1s = 3 \times 10^8$m so we get

$$A = 4.35 \times 10^{17} \cdot 3 \times 10^8 = 1.3 \times 10^{26} m$$

Since the conversion factor is $1m = 1.35 \times 10^{27} kg$ we get

$$A = 1.3 \times 10^{26} \cdot 1.35 \times 10^{27} = 1.74 \times 10^{53} kg$$

These are the required values. □

(c) The orbital speed of the earth.
Solution:
The mass of Earth is $M = 6 \times 10^{24} kg$ which with the conversion factor $1m = 1.35 \times 10^{27}$kg becomes

$M = 4.45 \times 10^{-3} m$ and the readius of earth (R) is $R = 6.4 \times 10^6 m$ and for our units $G = 1$ The orbital speed (v) is given by

$$v^2 = \frac{GM}{R} = \frac{4.45 \times 10^{-3} m}{6.4 \times 10^6 m} = 6.97 \times 10^{-10} m^0$$
$$v = 2.64 \times 10^{-5} m^0$$

Since the orbital speed is dimensionless, it has to have same value in kg unit also so

$$v = 2.64 \times 10^{-5} kg^0$$

These are the required values for orbital speed in each units. □

12.1.2. **(Schutz 1.3)** Draw t and x axes of the spacetime coordinates of an observer $\mathcal{O}$ and then draw:

(a) The world line $\mathcal{O}$'s clock at $x = 1$m.
(b) The world line of a particle moving with velocity $\frac{dx}{dt} = 0.1$, and which is at $x = 0.5$m and when $t = 0$.
(c) The $\bar{t}$ and $\bar{x}$ axes of an observer $\bar{\mathcal{O}}$ whio moves with velocity $v = 0.5$ in the positive x direction relative to $\mathcal{O}$ and whose origin $\bar{x} = \bar{t} = 0$ coincides with that of $\mathcal{O}$.
(d) The locus of events whose interval Δs^2 from origin is $-1m^2$.
(e) The locus of events whose interval Δs^2 from origin is $+1m^2$.
(f) The calibration ticks at one meter intervals along the $\bar{x}$ and $\bar{t}$ axes.

12.1.3. **(Schutz 2.1)** Given the numbers $\{A^0 = 5, A^1 = 0, A^2 = -1, A^3 = -6\}$, $\{B_0 = 0, B_1 = -2, B_2 = 4, B_3 = 0\}$, $\{C_{00} = 1, C_{01} = 0, C_{03} = 3, C_{30} = -1, C_{10} = 6, C_{11} = -2, C_{12} = -2, C_{13} = 0, C_{21} = 5, C_{22} = 2, C_{23} = -2, C_{20} = 4, C_{32} = -1, C_{32} = -3, C_{33} = 0\}$, find:

(a) $A^\alpha B_\alpha$
Solution:

$$A^\alpha B_\alpha = 5 * 0 + 0 * -2 + -1 * 4 + 6 * 0 = -4$$

□

(b) $A^\alpha C_{\alpha\beta}$ for all β
Solution:
for $\beta = 0$

$$A^\alpha C_{\alpha 0} = A^0 C_{00} + A^1 C_{10} + A^2 C_{20} + A^3 C_{30}$$
$$= 5 * 1 + 0 * 5 + -1 * 4 - 6 * -1 = 7$$

Similarly

$$A^\alpha C_{\alpha 1} = 0 + 0 + -5 + 6 = 1$$
$$A^\alpha C_{\alpha 2} = 10 + 0 + -2 + 18 = 26$$
$$A^\alpha C_{\alpha 3} = 15 + 0 + 3 + 0 = 18$$

□

(c) $A^\gamma C_{\gamma\sigma}$ for all σ
Solution:
This is same as the previous one because the dummy index is the only one different. □

(d) $A^\nu C_{\mu\nu}$ for all μ
Solution:

□

(e) $A^\alpha B_\beta$ for all α, β
(f) $A^i B_i$
(g) $A^j B_k$ for all j, k

12.1.4. **(Schutz 2.14)** The following matrix gives a Lorents transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$:

$$\begin{bmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{bmatrix}$$

(a) What is the velocity of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$?
(b) What is the inverse matrix to the given one?
(c) Find the components in $\mathcal{O}$ of a vector $\boldsymbol{A} \to (1, 2, 0, 0)$.

12.1.5. **(Schutz 2.22)**

(a) Find the energy, rest mass and three-veloity v of a particle whose four momentus has the components $(0, 1, 1, 0)kg$.
(b) The collision of two particles of four-momentum

$$\boldsymbol{p}_1 \underset{\mathcal{O}}{\longrightarrow} (3, -1, 0, 0)kg, \qquad \boldsymbol{p}_2 \underset{\mathcal{O}}{\longrightarrow} (2, 1, 1, 0)kg$$

results in the destruction fo the two particle and the production fo three new ones, two of which have four-mementa

$$\boldsymbol{p}_3 \underset{\mathcal{O}}{\longrightarrow} (1, 1, 0, 0)kg, \qquad \boldsymbol{p}_4 \underset{\mathcal{O}}{\longrightarrow} (1, -1/2, 0, 0)kg$$

Find the four-meomentum, energy, rest mass and three velocity of the third particle produced. Find the CM frame's three-velocity.

12.1.6. **(Schutz 2.30)** The four-velocity of a rocket ship is $\boldsymbol{U} \underset{\mathcal{O}}{\longrightarrow} (2, 1, 1, 1)$. It encounters a high-velocity cosmic ray whose mementum is $\boldsymbol{P} \underset{\mathcal{O}}{\longrightarrow} (300, 299, 0, 0) \times 10^{-27} kg$. Compute the energy of the cosmic ray as measured by the rocket ship's passengers, using each of the two following methods.

(a) Find the Lorentz transformation from $\mathcal{O}$ to the MCRF of the rocket ship, and use it to tranform the componetns of $\boldsymbol{P}$.
(b) Use eq 2.35
(c) Which method is quicker? Why?

12.2 Homework Two

12.2.1. A particle in Minkowski space travels along a trajectory:

$$x(\tau) = \alpha\tau^2$$
$$y(\tau) = \tau$$
$$z(\tau) = 0$$

(a) What are the spacelike components of the 4-velocity, U^i?
Solution:
The spacelike components of four velocity is

$$U^i = \frac{\partial x^i}{\partial \tau} = (2\alpha\tau, 1, 0)$$

□

(b) Using the relation $U \cdot U = -1$, compute U^0.
Solution:
The inner product of the four velocity vector $U^\mu = (U^0 U^1 U^2 U^3)$ is

$$\begin{aligned} U \cdot U = -(U^0)^2 + (U^1)^2 + (U^2)^2 + (U^3)^2 = -1 \\ \implies -(U^0)^2 + 4\alpha^2\tau^2 + 1 + 0 = -1 \\ \implies U^0 = \pm\sqrt{2 + (2\alpha\tau)^2} \end{aligned}$$

This is the timelike component of velocity four vector. □

(c) What is the 3-velocity of the particle as a function of τ?
Solution:
The spacelike components are given by

$$V^i = \frac{U^i}{U^0} = \left(\frac{2\alpha\tau}{\sqrt{2 + (2\alpha\tau)^2}}, \frac{1}{\sqrt{2 + (2\alpha\tau)^2}}, 0 \right)$$

□

12.2.2. **(Schutz 3.24)** Give the components of $\binom{2}{0}$ tensor $M^{\alpha\beta}$ as the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

find:

(a) the components of symmetric tensor $M^(\alpha\beta)$ and antisymmetric tensor $M^{[\alpha\beta]}$
Solution:
The symmetric tensor can be written as

$$M^{(\alpha\beta)} = \frac{1}{2}\left(M^{\alpha\beta} + M^{\beta\alpha}\right)$$

When the indices are switched the elements of the tensor are

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Using this we get the symmetric form

$$M^{(\alpha\beta)} = \begin{bmatrix} 0 & 1 & 1 & 1/2 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1/2 \\ 1/2 & 1 & -1/2 & 0 \end{bmatrix}$$

Similarly the anti symmetric tensor is

$$M^{[\alpha\beta]} = \begin{bmatrix} 0 & 0 & -1 & -1/2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3/2 \\ 1/2 & -1 & -3/2 & 0 \end{bmatrix}$$

These are the required matrices. □

(b) the components of ${M^\alpha}_\beta$
Solution:
This can be written with the metric tensor as

$$
{M^\alpha}_\beta = g_{\sigma\beta} M^{\alpha\sigma} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}
$$

□

(c) the components of M_α^β
Solution:
This can be written with th metric as

$$
{M_\alpha}^\beta = g_{\alpha\sigma} M^{\sigma\beta} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}
$$

□

(d) the components of $M_{\alpha\beta}$
Solution:
The previous tensor can be used to calculate this

$$
M_{\alpha\beta} = g_{\sigma\beta} {M_\alpha}^\sigma = \begin{bmatrix} 0 & -1 & & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}
$$

□

12.2.3. **(Schutz 3.30)** In some $\mathcal{O}$, the vector U and D have the components

$$
\begin{aligned}
U &\to (1+t^2, t^2, \sqrt{2}t, 0) \\
D &\to (x, 5tx, \sqrt{2}t, 0)
\end{aligned}
$$

and the scalar ρ has the value

$$
\rho = x^2 + t^2 - y^2
$$

(a) Find $U \cdot U$, $U \cdot D$, $D \cdot D$. Is U suitable as four-velocity field? Is D?
Solution:
The components of U_μ are $U_\mu = (-(1+t^2), t^2, \sqrt{2}t, 0)$ and the components of D_μ are $D_m u = (-x, 5tx, \sqrt{2}t, 0)$ so the dot products are

$$
\begin{aligned}
U \cdot U = U^\mu U_m u &= (-(1+t^2)^2 + t^4 + 2t^2 + 0) = -1 - 2t^2 - t^4 + t^4 + 2t^2 = -1 \\
D \cdot D = D^\mu D_\mu &= (-x^2 + 25t^2x^2 + 2t^2 + 0) = x^2(25t^2 - 1) + 2t^2 \\
U \cdot D = U^\mu D_\mu &= -x(1+t^2) + 5t^3x + 2t^2 = x(5t^3 - t^2 - 1) + 2t^2
\end{aligned}
$$

Since the inner product of U with itself is -1 its is suitable for a four velocity while D is not (except possibly for fixed values of x and t). □

(b) Find the spatial velocity v of a particle whose four-velocity is U,, for arbitrary t. What happens to it in the limits $t \to 0$ and $t \to \infty$?
Solution:

$$v^i = \frac{U^i}{U^0} = \left(\frac{t^2}{1+t^2}, \frac{\sqrt{2}t}{1+t^2}, 0 \right)$$

In the limit $t \to \infty$ we get $\boldsymbol{v} = (1, 0, 0)$ and in the limit $t \to 0$ we get $\boldsymbol{v} = (0, 0, 0)$ □

(c) Find U_α for all α
Solution:
With the Minkowski metric the values of U_α is $U_\alpha = (-(1+t)^2, t^2, \sqrt{2}t, 0)$ □

(d) Find $U^\alpha{}_{,\beta}$ for all α, β
Solution:
The vales are

$$U^\alpha{}_{,\beta} = \frac{\partial U^\alpha}{\partial x^\beta} = \begin{bmatrix} 2t & 0 & 0 & 0 \\ 2t & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

(e) Show that $U^\alpha_\alpha{}_{,\beta} = 0$ for all β. Show that $U^\alpha U_{\alpha,\beta} = 0$ for all β.
Solution:
For various values of β $U_\alpha U^\alpha_{,\beta}$ is

$$\beta = 0 :: U_\alpha U^\alpha_{,0} = \frac{\partial}{\partial t} \left(-(1+t^2)^2 + t^4 + 2t \right) = -2(1+t^2) \cdot 2t + 4t^3 + 4t = 0$$
$$\beta = 1 :: U_\alpha U^\alpha_{,1} = \frac{\partial}{\partial x} \left(-(1+t^2)^2 + t^4 + 2t \right) = 0$$
$$\beta = 2 :: U_\alpha U^\alpha_{,2} = \frac{\partial}{\partial y} \left(-(1+t^2)^2 + t^4 + 2t \right) = 0$$
$$\beta = 3 :: U_\alpha U^\alpha_{,3} = \frac{\partial}{\partial z} \left(-(1+t^2)^2 + t^4 + 2t \right) = 0$$

We have $U^\alpha U_\alpha$ is the inner product of $U \cdot U$ and so $U \cdot U = U^\alpha U_\alpha = U_\alpha U^\alpha$ so the expression

$$U^\alpha U_{\alpha,\beta} = (U_\alpha U^\alpha)_{,\beta} = 0, \forall \beta$$

□

(f) Find $D^\beta{}_{,\beta}$
Solution:
It is simply the divergence of vector D so we get

$$D^\beta_{,\beta} = \frac{\partial x}{\partial t} + \frac{\partial 5tx}{\partial x} + \frac{\partial \sqrt{2}t}{\partial y} + \frac{\partial 0}{\partial z} = 5t$$

□

(g) Find $(U^\alpha D^\beta)_{,\beta}$ for all α.
Solution:

The components of tensor $U^\alpha D^\beta$ are

$$U^\alpha D^\beta = \begin{bmatrix} (1+t^2)x & 5tx(1+t^2) & \sqrt{2}t(1+t^2) & 0 \\ t^2x & 5t^3x & \sqrt{2}t^3 & 0 \\ \sqrt{2}tx & 5\sqrt{2}t^2x & 2t^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now the derivatives $(U^\alpha D^\beta)_{,\beta}$ has the components

$$\begin{aligned} &\alpha = 0 : 2tx + 5t(1+t^2) + 0 + 0 = 2tx + 5t(1+t^2) \\ &\alpha - 1 : 2tx + 5t^3 + 0 + 0 = 2tx + 5t^3 \\ &\alpha = 2 : \sqrt{2}x + 5\sqrt{2}t^2 + 0 + 0 = \sqrt{2}x + 5\sqrt{2}t^2 \\ &\alpha = 3 : 0 \end{aligned}$$

So the components are $(U^\alpha D^\beta)_{,\beta} = (2tx + 5t(1+t^2), 2tx + 5t^3, \sqrt{2}x + 5\sqrt{2}t^2)$. □

(h) Find $U_\alpha (U^\alpha D^\beta)_{,\beta}$ and compare result.
Solution:
We have the components of $U_\alpha = (-(1+t^2), t^2, \sqrt{2}t, 0)$ and we have obtained

$$M^\alpha = (U^\alpha D^\beta)_{,\beta} = (2tx + 5t(1+t^2), 2tx + 5t^3, \sqrt{2}x + 5\sqrt{2}t^2)$$

$$\begin{aligned} U_\alpha (U^\alpha D^\beta)_{,\beta} &= U_\alpha M^\alpha \\ &= (-(1+t^2)(2tx + 5t(1+t^2)) + t^2(2tx + 5t^3) + \sqrt{2}t(\sqrt{2}xt + 5\sqrt{2}t^2)) \\ &= -5t \end{aligned}$$

We see that this is equal to $-D^\beta_{,\beta}$ and using the fact that $U_\alpha U^\alpha = -1$ we can rewrite

$$U_\alpha (U^\alpha D^\beta)_{,\beta} = -D^\beta_{,\beta} = (U_\alpha U^\alpha) D^\beta_{,\beta}$$

This shows that the associative property in tensors hold. □

(i) Find $\rho_{,\alpha}$ for all α. Find $\rho^{,\alpha}$ for all α
Solution:
The components are

$$\rho_{,\alpha} = \left(\frac{\partial \rho}{\partial t}, \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z} \right) = (2t, 2x, -2y, 0)$$

The raised version is

$$\rho^{,\beta} = (-2t, 2x, -2y, 0)$$

□

12.2.4. **(Schuts 4.17)** We have defined $a^\mu = U^\mu_{,\beta} U^\beta$. Go to the non-relativistic limit and show that

$$a^i = \dot{v}^i + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v^i$$

Solution:
Writing out the components of the above expression we get

$$a^\mu = \frac{\partial U^\mu}{\partial x^0} U^0 + \frac{\partial U^i}{\partial x^1} U^1 + \frac{\partial U^i}{\partial x^2} U^2 + \frac{\partial U^i}{\partial x^3} U^3$$

The spatial components are

$$a^i = \frac{\partial U^i}{\partial x^0}U^0 + \frac{\partial U^i}{\partial x^1}U^1 + \frac{\partial U^i}{\partial x^2}U^2 + \frac{\partial U^i}{\partial x^3}U^3$$

In the non relativistic limit $U^0 = 1$ and $U^i = v^i$ where v^i is the component of velocity so we obtain

$$a^i = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x}v^x + \frac{\partial v^i}{\partial y}v^y + \frac{\partial v^i}{\partial z}v^z$$

This expression can be rearranged into

$$a^i = \dot{v}^i + (v^x\hat{\boldsymbol{i}} + v^y\hat{\boldsymbol{j}} + v^z\hat{\boldsymbol{k}}) \cdot \left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right) v^i$$

Since the nabla operator is the middle term in above expression we get

$$a^i = \dot{v}^i + (\boldsymbol{v} \cdot \boldsymbol{\nabla})v^i$$

This is the required expression. □

12.2.5. Consider a stationary, ideal fluid of the form:

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

For the moment, you should assume that the stress-energy tensor is constant in time and throughout space

(a) Compute the stress energy tensor $T^{\bar{\mu}\bar{\nu}}$ in a frame moving at a speed, v with respect to th frame along the x-axis.
Solution:
The transformation matrix is

$$\Lambda^{\bar{\mu}}_{\mu} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The components of the transformed tensor are

$$\begin{aligned} T^{\bar{\mu}\bar{\nu}} &= \Lambda^{\bar{\mu}}_{\mu}\left[\Lambda^{\bar{\nu}}_{\nu}T^{\mu\nu}\right] \\ &= \Lambda^{\bar{\mu}}_{\mu}\left[\Lambda^{\bar{\nu}}_{0}T^{\mu 0} + \Lambda^{\bar{\nu}}_{1}T^{\mu 1} + \Lambda^{\bar{\nu}}_{2}T^{\mu 2} + \Lambda^{\bar{\nu}}_{3}T^{\mu 3}\right] \end{aligned}$$

Since the off diagonal elements of $T^{\mu\nu}$ are all zeros we get zeros for all j

$$T^{\bar{\mu}\bar{\nu}} = \Lambda^{\bar{\mu}}_{0}\left[\Lambda^{\bar{\nu}}_{0}T^{00}\right] + \Lambda^{\bar{\mu}}_{1}\left[\Lambda^{\bar{\nu}}_{1}T^{11}\right] + \Lambda^{\bar{\mu}}_{2}\left[\Lambda^{\bar{\nu}}_{2}T^{22}\right] + \Lambda^{\bar{\mu}}_{3}\left[\Lambda^{\bar{\nu}}_{3}T^{33}\right]$$

So we get the transformed tensor as

$$\begin{bmatrix} \gamma^2\rho + \gamma^2 v^2 P & \gamma^2 v\rho + \gamma^2 vP & 0 & 0 \\ \gamma^2 v\rho + \gamma^2 vP & \gamma^2 v^2 P + \gamma^2\rho & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} = \begin{bmatrix} \gamma^2(\rho + v^2 P) & \gamma^2 v(\rho + P) & 0 & 0 \\ \gamma^2 v(\rho + P) & \gamma^2(v^2\rho + P) & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

This is the required transformed tensor. □

(b) Suppose the pressure is a fixed ratio to the density. Compute the stress energy tensor in the moving frame for i) P = 0 (dust), ii) P= $1/3\rho$ (radiation) iii) $P = -\rho$ (cosmological constant).
Solution:
for P = 0 we get

$$\begin{bmatrix} \gamma^2\rho & \gamma^2 v\rho & 0 & 0 \\ \gamma^2 v\rho & \gamma^2 v^2\rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for P = 1/3 ρ we get

$$\begin{bmatrix} \gamma^2(\rho + v^2\frac{1}{3}\rho) & \gamma^2 v(\frac{4}{3}\rho) & 0 & 0 \\ \gamma^2 v(\frac{4}{3}\rho) & \gamma^2(v^2\rho + \frac{1}{3}\rho) & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{bmatrix}$$

for $P = -\rho$ we get

$$\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 \\ 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & -\rho \end{bmatrix}$$

These are the transformed tensor. □

12.3 Homework Three

12.3.1. In a flat space, the metric in spherical coordinates , r, θ, ϕ is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

(a) Compute all non-zero Christoffel symbols for this system.
(b) Compute the divergence $V^{\alpha}{}_{;\alpha}$

12.3.2. Consider a vectro in 2-d space:

$$\boldsymbol{v} = \hat{\boldsymbol{i}}$$

starting at $r = 1, \theta = 0$, and moving aroudn the unit circle with constatn $r = 1$, but varying θ. The assumption is that the vector itself should not vary.

Write, and solve a differential equation describing the changes in the components of $\boldsymbol{v}$ as you parallel-transport it around the unit circle.

12.3.3. **(Schutz 5.14)** For the tensor whose polar components are $A^{rr} = r^2, A^{r\theta} = r\sin\theta, A^{\theta r} = r\cos\theta, A^{\theta\theta} = \tan\theta$, compute

$$\nabla_\beta A^{\mu\nu} = A^{\mu\nu}{}_{,\beta} + A^{\alpha\nu}\Gamma^{\mu}{}_{\alpha\beta} + A^{\mu\alpha}\Gamma^{\nu}{}_{\alpha\beta}$$

in polars for all possible indices.

12.3.4. **(Schutz 7.3)** Calculate all the Christoffel symbols for the metric,

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)\left(dx^2 + dy^2 + dz^2\right)$$

, to first order in ϕ. Assume ϕ is a general function of t, x, y and z.

12.3.5. A cosmic string is a theoretical construct whcih is infinitely long, and has a mass density per unit length λ. The coordinates describibg the spacetime surrounding a cosmic string are

$$x^\mu = \begin{pmatrix} t \\ R \\ \phi \\ z \end{pmatrix}$$

and which has a metric:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R^2(1-4\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Compute the volume element, dV, near the cosmic string.
(b) Compute all non-zero Christoeffl symbols.
(c) Compute the distance between two points separated by $dx^\mu = dR$, and all other coordinates equal to zero. From that, cmptute the distance from the string itself out to t distance $R = 1$
(d) Compute the distance between two points, each $R = 1$ from the string separated by an angle $d\phi$ (with all other $dx^\mu = 0$) Using that, what is the total distance traersed by a particle covering a circular orbit $R = 1$ around the cosmic string?
(e) Compare (12.3.5c) and (12.3.5d) in the context of the normal relationship between radius and circumference. That is, does $C = 2\pi r$? if not, what should it be replaced with?

12.4 Homework Four

12.4.1. **(Schutz 6.29)** In polar coordinates, calculate the Riemann curvature tensor of the sphere of unit radius whose metric is $g_{\theta\theta} = r^2, g_{\phi\phi} = r^2 \sin^2\theta, g_{\theta\phi} = 0$.
Solution:
The metric for polar coordinate on the surface of unit sphere is

$$\begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

The christoffel symbols are given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma}{}_{,\rho} + g_{\rho\sigma}{}_{,\nu} - g_{\nu\rho}{}_{,\sigma} \right)$$

The only non zero derivative of metric is with respect to θ so we get

$$\Gamma^\theta{}_{\phi\phi} = \frac{1}{2} g^{\theta\theta} \left(-g_{\phi\phi,\theta}\right) = -\frac{1}{2}\sin 2\theta$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

And the Riemann tensor is given by

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} - \Gamma^\alpha_{\beta\mu}{}_{,\nu} + \Gamma^\alpha_{\beta\nu}{}_{,\mu}$$
$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda}\left(\Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\beta\mu} - \Gamma^\lambda_{\beta\mu}{}_{,\nu} + \Gamma^\lambda_{\beta\nu}{}_{,\mu} \right)$$

Calculating

$$\begin{aligned}
R_{\phi\theta\phi\theta} &= g_{\phi\phi}\left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{\theta\theta} - \Gamma^{\phi}_{\sigma\theta}\Gamma^{\sigma}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi,\theta} + \Gamma^{\phi}_{\theta\theta,\phi}\right)\\
&= \sin^2\theta\left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{\theta\theta} - \Gamma^{\phi}_{\sigma\theta}\Gamma^{\sigma}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi,\theta} + \Gamma^{\phi}_{\theta\theta,\phi}\right)\\
&= \sin^2\theta\left(-\Gamma^{\phi}_{\phi\theta}\Gamma^{\phi}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi,\theta}\right)\\
&= \sin^2\theta\left(-\frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\sin^2\theta}\right)\\
&= \sin^2\theta
\end{aligned}$$

Now we can permute the coordinate with the symmetry property to obtain

$$R_{\phi\theta\theta\phi} = -\sin^2\theta \qquad R_{\theta\phi\phi\theta} = -\sin^2\theta \qquad R_{\theta\phi\theta\phi} = \sin^2\theta$$

These are the non zero components of Riemann tensor. □

12.4.2. **(Schutz 6.30)** Calculate the Riemann curvature tensor of the cylinder.
Solution:
The line element ins the cylindrical coordinate system is

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

So the metric in is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right)$$

The only non zero derivative of metric is with respect to θ so we get

$$\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}g^{rr}\left(-g_{\phi\phi,r}\right) = -r$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}$$

And the Riemann tensor is given by

$$\begin{aligned}
R^{\alpha}{}_{\beta\mu\nu} &= \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\beta\nu,\mu}\\
R_{\alpha\beta\mu\nu} &= g_{\alpha\lambda}\left(\Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\lambda}_{\beta\mu,\nu} + \Gamma^{\lambda}_{\beta\nu,\mu}\right)
\end{aligned}$$

Calculating

$$\begin{aligned}
R_{\phi r\phi r} &= g_{\phi\phi}\left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{rr} - \Gamma^{\phi}_{\sigma r}\Gamma^{\sigma}_{r\phi} - \Gamma^{\phi}_{r\phi,r} + \Gamma^{\phi}_{rr,\phi}\right)\\
&= r^2\left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{rr} - \Gamma^{\phi}_{\sigma r}\Gamma^{\sigma}_{r\phi} - \Gamma^{\phi}_{r\phi,r} + \Gamma^{\phi}_{rr,\phi}\right)\\
&= r^2\left(-\Gamma^{\phi}_{\phi r}\Gamma^{\phi}_{r\phi} - \Gamma^{\phi}_{r\phi,r}\right)\\
&= r^2\left(-\frac{1}{r^2} + \frac{1}{r^2}\right)\\
&= 0
\end{aligned}$$

Now we can permute the coordinates and with symmetry all the rest are zero too.

$$R_{\phi rr\phi} = 0 \qquad R_{r\phi\phi r} = 0 \qquad R_{r\phi r\phi} = 0$$

So all the components of Riemann tensor are zero, showing that the surface of cylinder is a flat surface.
□

12.4.3. One way of describing the metric of a flat, homogeneous, expanding universe is:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix}$$

where $a(t)$ is a function of time only, and the coordinates are

$$x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

(a) Compute all non vanishing terms of the Riemann Tensor.
Solution:

The Christoffel symbols are given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma}{}_{,\rho} + g_{\rho\sigma}{}_{,\nu} - g_{\nu\rho}{}_{,\sigma} \right)$$

The only non zero derivative of metric is with respect to t so we get

$$\Gamma^t{}_{xx} = \frac{1}{2} g^{tt} \left(-g_{xx,t} \right) = a\dot{a}$$

These are true for y and z coordinates.

$$\Gamma^t{}_{yy} = \frac{1}{2} g^{tt} \left(-g_{yy,t} \right) = a\dot{a} \qquad \Gamma^t{}_{zz} = \frac{1}{2} g^{tt} \left(-g_{zz,t} \right) = a\dot{a}$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^x{}_{tx} = \Gamma^x{}_{xt} = \frac{\dot{a}}{a}$$

These are also true for y and z.

$$\Gamma^y{}_{ty} = \Gamma^y{}_{yt} = \frac{\dot{a}}{a} \qquad \Gamma^z{}_{tz} = \Gamma^z{}_{zt} = \frac{\dot{a}}{a}$$

And the Riemann tensor is given by

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu} &= \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} - \Gamma^\alpha_{\beta\mu}{}_{,\nu} + \Gamma^\alpha_{\beta\nu}{}_{,\mu} \\ R_{\alpha\beta\mu\nu} &= g_{\alpha\lambda} \left(\Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\beta\mu} - \Gamma^\lambda_{\beta\mu}{}_{,\nu} + \Gamma^\lambda_{\beta\nu}{}_{,\mu} \right) \end{aligned}$$

Calculating

$$\begin{aligned} R_{xtxt} &= g_{xx} \left(\Gamma^x_{\sigma x}\Gamma^\sigma_{tt} - \Gamma^x_{\sigma t}\Gamma^\sigma_{tx} - \Gamma^x_{tx,t} + \Gamma^x_{tt,x} \right) \\ &= a^2 \left(\Gamma^x_{\sigma x}\Gamma^\sigma_{tt} - \Gamma^x_{\sigma t}\Gamma^\sigma_{tx} - \Gamma^x_{tx,t} + \Gamma^x_{tt,x} \right) \\ &= a^2 \left(-\Gamma^x_{xt}\Gamma^x_{tx} - \Gamma^x_{tx,t} \right) \\ &= a^2 \left(-\frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right) \\ &= a\ddot{a} \end{aligned}$$

Now we can permute the coordinate with the symmetry property to obtain

$$R_{xttx} = -a\ddot{a} \qquad R_{txxt} = -a\ddot{a} \qquad R_{txtx} = a\ddot{a}$$

Similarly the rest of the values can be calculated as

$$R_{yxxy} = -a^2\dot{a}^2$$

The rest of them can be obtained by permuting the index using the (anti-)symmetry property.

$$R_{zxxz} = R_{zyyz} = R_{yzzy} = R_{xzzx} = R_{xyyx} = -a^2\dot{a}$$

□

(b) Compute all Non-vanishing terms of the Ricci Tensor.
Solution:
The raised version of Riemann tensor is

$$R^{\alpha}{}_{\beta\gamma\mu} = g^{\alpha\sigma}R_{\sigma\beta\gamma\mu}$$

The first index non vanishing term is

$$\begin{aligned} R^{x}_{ttx} &= g^{tt}\cancelto{0}{R_{tttx}} + g^{xx}R_{xttx} + g^{yy}\cancelto{0}{R_{yttx}} + g^{zz}\cancelto{0}{R_{zttx}} \\ &= a^{-2}\left(-a\ddot{a}\right) = -\frac{\ddot{a}}{a} \end{aligned}$$

Using the symmetry property and the elements of metric we get the rest of components of Riemann tensor as

$$\begin{aligned} R^{x}_{ttx} &= -\frac{\ddot{a}}{a} \qquad & R^{x}_{txt} &= \frac{\ddot{a}}{a} \\ R^{y}_{tty} &= -\frac{\ddot{a}}{a} \qquad & R^{y}_{tyt} &= \frac{\ddot{a}}{a} \\ R^{z}_{ttz} &= -\frac{\ddot{a}}{a} \qquad & R^{z}_{tzt} &= \frac{\ddot{a}}{a} \end{aligned}$$

Now the components of Ricci tensor in terms of elements of Riemann Rensor are

$$R_{\alpha\beta} = g^{\mu\nu}R^{\nu}_{\alpha\mu\beta}$$

Specifically for R_{tt} we get

$$\begin{aligned} R_{tt} &= g^{tt}\cancelto{0}{R^{t}_{ttt}} + g^{xx}R^{x}_{txt} + g^{yy}R^{y}_{tyt} + g^{zz}R^{z}_{tzt} \\ &= -a^{-2}\ddot{a}a - a^{-2}\ddot{a}a - a^{-2}\ddot{a}a \\ &= -3\ddot{a}/a \end{aligned}$$

Similarly rest of the components can be calculated. They are

$$R_{xx} = R_{yy} = R_{zz} = a\ddot{a} + 2\dot{a}^2$$

These are the components of Ricci tensor □

(c) Compute Einstein Tensor.
Solution:
The components of Einstein tensor are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{12.1}$$

The Ricci scalar can be calculated by contracting the Ricci tensor as

$$R = R^t_t + R^x_x + R^y_y + R^z_z = 6\frac{a\ddot{a} + \dot{a}^2}{a^2} \tag{12.2}$$

Now the Einstein tensor simply is the substitution (12.3) into the (12.4). The first component of this tensor is

$$G_{tt} = R_{tt} - \frac{1}{2}g_{tt}R = -3\frac{\ddot{a}}{a} + \frac{1}{2}\frac{6\left(a\ddot{a} + \dot{a}^2\right)}{a^2} = 3\frac{\dot{a}^2}{a^2}$$

Similarly the rest of the components can be calculated.

$$G_{xx} = R_{xx} - \frac{1}{2}g_{xx}R = a\ddot{a} + 2\dot{a}^2 - \frac{1}{2}a^2 \cdot \frac{6(a\ddot{a} + \dot{a}^2)}{a^2} = -2a\ddot{a} - \dot{a}^2$$

$$G_{xx} = G_{yy} = G_{zz} = -2a\ddot{a} - \dot{a}^2$$

The raised version of Einstein tensor similarly are[1].

$$G^{tt} = \frac{3}{2}\frac{\dot{a}^2}{a^2} \qquad G^{xx} = G^{yy} = G^{zz} = -\frac{\dot{a}^2 + 2a\ddot{a}}{a^4}$$

These are the required components of Einstein tensor. □

12.4.4. **(Schutz 6.35)** Compute 20 independent components of $R_{\alpha\beta\mu\nu}$for a manifold with line element $ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, where Φ and Λ are arbitrary functions fo the coordinate r alone.
Solution:
Writing down the metric from the given expression for line element

$$g_{tt} = -e^{2\Phi}; \qquad g_{rr} = e^{2\Lambda}; \qquad g_{\theta\theta} = r^2; \qquad g_{\phi\phi} = r^2\sin^2\theta$$

The inverse metric is

$$g^{tt} = -e^{-2\Phi}; \qquad g^{rr} = e^{-2\Lambda}; \qquad g^{\theta\theta} = \frac{1}{r^2}; \qquad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}$$

The Christoffel symbols can be calculated by the expression

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right)$$

Evaluating the these we get

$$\begin{aligned}
&\Gamma^t_{rt} = \Gamma^t_{tr} = \Phi_{,r}\\
&\Gamma^r_{\phi\phi} = -re^{-2\Lambda}\sin^2\theta \qquad \Gamma^r_{tt} = -re^{-2\Lambda+2\Phi}\Phi_{,r} \qquad \Gamma^r_{rr} = \Lambda_{,r} \qquad \Gamma^r_{\theta\theta} = -re^{-2\Lambda}\\
&\Gamma^\theta_{\phi\phi} = \frac{1}{2}\sin 2\theta \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}\\
&\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

The Riemann tensor is given by

$$\begin{aligned}
R^\alpha_{\beta\mu\nu} &= \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}\\
R_{\alpha\beta\mu\nu} &= g_{\lambda\alpha}(\Gamma^\lambda_{\beta\mu,\nu} + \Gamma^\lambda_{\beta\nu,\mu} - \Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\beta\mu})
\end{aligned}$$

[1]This was solved mostly using Cadabra. https://www.physics.drexel.edu/~pgautam/courses/PHYS631/einstein-tensor-expanding-universe.html

Explicitly for R_{trtr} we get

$$\begin{aligned} R_{trtr} &= g_{tt} R^t_{rtr} \\ &= -e^{2\Phi}\left[\Gamma^r_{rt,r} + \cancelto{0}{\Gamma^r_{rr,t}} + \Gamma^r_{\sigma r}\Gamma^\sigma_{rr} - \Gamma^r_{\sigma r}\Gamma^\sigma_{rt}\right] \\ &= -e^{2\Phi}\left[\Phi_{,rr} + \Gamma^r_{rr}\Gamma^r_{rr} - \Gamma^r_{rr}\Gamma^r_{rt}\right] \\ &= -e^{2\Phi}\left[\Phi_{,rr} + (\Phi_{,r})^2 - \Phi_{,r}\Lambda_{,r}\right] \end{aligned}$$

The rest of the components can be similarly calculated [2]

$$\begin{aligned} R_{trrt} &= ((\Lambda_{,r} - \Phi_{,r})\Phi_{,r} - \Phi_{,rr})\, e^{2\Phi} \\ R_{trtr} &= (-(\Lambda_{,r} - \Phi_{,r})\Phi_{,r} + \Phi_{,rr})\, e^{2\Phi} \end{aligned}$$

$$\begin{aligned} R_{t\phi t\phi} &= re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r} \\ R_{t\theta t\theta} &= re^{-2\Lambda+2\Phi}\Phi_{,r} \\ R_{t\phi\phi t} &= -re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r} \\ R_{t\theta\theta t} &= -re^{-2\Lambda+2\Phi}\Phi_{,r} \\ R_{r\phi r\phi} &= r\sin^2\theta\Lambda_{,r} \\ R_{r\theta r\theta} &= r\Lambda_{,r} \\ R_{rtrt} &= (-(\Lambda_{,r} - \Phi_{,r})\Phi_{,r} + \Phi_{,rr})\, e^{2\Phi} \\ R_{r\phi\phi r} &= -r\sin^2\theta\Lambda_{,r} \\ R_{r\theta\theta r} &= -r\Lambda_{,r} \\ R_{rttr} &= ((\Lambda_{,r} - \Phi_{,r})\Phi_{,r} - \Phi_{,rr})\, e^{2\Phi} \\ R_{\theta rr\theta} &= -r\Lambda_{,r} \\ R_{\theta r\theta r} &= r\Lambda_{,r} \\ R_{\theta\phi\theta\phi} &= \frac{1}{2}r^2\left(e^{2\Lambda}\sin(2\theta)(\tan\theta)^{-1} - 2e^{2\Lambda}\cos(2\theta) + \cos(2\theta) - 1\right)e^{-2\Lambda} \\ R_{\theta t\theta t} &= re^{-2\Lambda+2\Phi}\Phi_{,r} \\ R_{\theta\phi\phi\theta} &= r^2\left(1 - e^{2\Lambda}\right)e^{-2\Lambda}\sin^2\theta \\ R_{\theta tt\theta} &= -re^{-2\Lambda+2\Phi}\Phi_{,r} \\ R_{\phi rr\phi} &= -r\sin^2\theta\Lambda_{,r} \\ R_{\phi\theta\theta\phi} &= r^2\left(1 - e^{2\Lambda}\right)e^{-2\Lambda}\sin^2\theta \\ R_{\phi r\phi r} &= r\sin^2\theta\Lambda_{,r} \\ R_{\phi\theta\phi\theta} &= r^2\left(e^{2\Lambda} - 1\right)e^{-2\Lambda}\sin^2\theta \\ R_{\phi t\phi t} &= re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r} \\ R_{\phi tt\phi} &= -re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r} \end{aligned}$$

These are the non zero components of Riemann tensor. □

12.4.5. **(Schutz 7.7)** Consider the following four different metrics, as given by their line elements:

i. $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$;

ii. $ds^2 = -(1 - 2M/r)\,dt^2 + (1 - 2M/r)^-1dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ where M is a constant.

[2] I did this using Cadabra. The detail of this exercise is at https://www.physics.drexel.edu/~pgautam/courses/PHYS631/HW4Schutz6.35.html

iii. $ds^2 = -dt^2 + R^2(t)\left[(1-kr^2)^-1dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$, where k is a constant and $R(t)$ is an arbitrary function of t alone.

(a) For each metric find as many conserved components p_α of a freely falling particle's four momentum as possible.
Solution:
The rate of change of momentum is given by

$$m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} = \frac{1}{2}g_{\mu\alpha,\beta}p^\nu p^\alpha$$

The momentum p_β is conserved when $g_{\mu\alpha,\beta} = 0$. From the given metric the conserved quantities are

$$\begin{aligned} \text{for } i. : &\quad p_t, p_x, p_y, p_z \\ \text{for } ii. : &\quad p_t, p_\phi \\ \text{for } iii. : &\quad p_\phi \end{aligned}$$

□

(b) Write i. in the form

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

From this argue that ii. iii. are spherically symmetric. Does this increase the number of conserved components of p_α?
Solution:
The coordinate transformation from Cartesian to polar is

$$\begin{aligned} x &= r\sin\theta\cos\phi &&\implies dx = \sin\theta + \cos\phi dr + +r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi \\ y &= r\sin\theta\sin\phi &&\implies dy = \sin\theta + \sin\phi dr + +r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi \\ z &= r\cos\theta &&\implies dz\cos\theta dr = -\sin\theta d\theta \end{aligned}$$

Substituting these in the line element we get

$$\begin{aligned} dl^2 =& - dt^2 + dr^2(\sin^2\theta\sin^2\phi\sin^2\theta\cos^2\phi + \cos^2) + \\ &+ d\theta^2\left(r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta\right) \\ &+ d\phi^2\left(r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi\right) \\ &= -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned}$$

This is the required transformation in spherical form. □

(c) It can be shown that for ii. and iii. a geodesic that begins with $\theta = \frac{\pi}{2}$ and $p^\theta = 0$- i.e., one which begins tangent to the equatorial plane- always has $\theta = \frac{\pi}{2}$ and $p^\theta = 0$. For these cases use the equation $\boldsymbol{p}\cdot\boldsymbol{p} = -m^2$ to solve for p^r in terms of m, other conserved quantities, and known functions of position.
Solution:
Expanding the relation $\boldsymbol{p}\cdot\boldsymbol{p} = -m^2$ we get

$$-m^2 = g_{tt}\left(p^t\right)^2 + g_{rr}\left(p^r\right)^2 + g_{\theta\theta}\left(p^\theta\right)^2 + g_{\phi\phi}\left(p^\phi\right)^2$$

Given $\theta = \pi/2$ and $p^\theta = 0$ we get

$$p^r = \sqrt{\frac{-m^2 - g_{tt}\left(p^t\right)^2 + g_{\phi\phi}\left(p^\phi\right)^2}{g_{rr}}}$$

Since p^t and p^ϕ are conserved substituting the corresponding metric values $g_{\alpha\beta}$ gives the quantity p^r

$$\text{for } ii.; \qquad p^r = \sqrt{\frac{-m^2 + \frac{1-2M}{r}\left(p^t\right)^2 + r^2\left(p^\phi\right)^2}{1 - 2M/r}}$$

$$\text{for } iii.; \qquad p^r = \sqrt{\frac{1-kr^2}{R^2(t)}\left(-m^2 + \left(p^t\right)^2 + (R(t)r)^2\left(p^\phi\right)^2\right)}$$

These are the required expression for p^r in terms of conserved quantities. □

(d) For iii., spherical symmetry implies that if a geodesic begins with $p^\theta = p^\phi = 0$, these remain zero. Use this to show that when $k = 0$, p_r is a conserved quantity.
Solution:
The rate of change of momentum is given by

$$m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} = \frac{1}{2}g_{\mu\alpha,\beta}p^\nu p^\alpha$$
$$m\frac{\mathrm{d}p_r}{\mathrm{d}\tau} = \frac{1}{2}\left(g_{tt,r}(p^t)^2 + g_{rr,r}(p^r)^2 + g_{\theta\theta,r}(p^\theta)^2 + g_{\phi\phi}(p^\phi)^2\right)$$

But for $k = 0$, $g_{rr,r} = 0$ and $g_{tt,r} = 0$ and given $p^\theta = p^\phi = 0$ we get

$$m\frac{\mathrm{d}p_r}{\mathrm{d}\tau} = 0$$

This proves that p_r is a conserved quantity. □

12.4.6. What fractional energy does a photon lose if it goes from the surface of the earth to deep space?
Solution:
When the photon goes from the surface of earth to outer space, it must lose the gravitational potential energy that is has near the surface of earth. So the photon must lose this energy. For photon

$$(U^0)^2 g_{00} = -1$$

On surface of earth with weak field limit

$$g_{00} = -(1 - 2\phi)$$

So near the surface of earth

$$U^0 \simeq 1 + \phi$$

In far space metric Minkowski $g_{00} = -1$ so in far space

$$U^0 = 1$$

So ratio of energy

$$\frac{1}{1+\phi} = 1 - \phi$$

So change in energy is $\sim \phi$ On the surface of earth the gravitational potential is

$$\phi = -\frac{GM}{c^2 r} = -\frac{6.672 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6 \times 9 \times 10^{16}} \approx 7 \times 10^{-10}$$

So the photon must lose this energy fractionally. □

12.5 Homework Five

12.5.1. Consider a 1+1 dimensional space (t, x) with the metric:

$$g_{\mu\nu} = \begin{pmatrix} -e^{kx} & 0 \\ 0 & 1 \end{pmatrix}$$

where k is a dimensional constant.

(a) This metric has a stress-energy source which is (potentially) non-zero. Knowing nothing else, what is the scaling of the density ρ in terms of k?
Solution:
Since the exponent in the metric has to be dimensionless the dimension of k is

$$[k] = [L] = [M]$$

The dimension of density is

$$[\rho] = \frac{[M]}{[L]^3} = [L]^2$$

From these two expressions

$$\rho \sim k^2$$

So, in terms of dimension only the density has to scale as the square of k. □

(b) Compute all non-zero Christoffel symbols.
Solution:
The non zero derivative of the metric is in terms of x only and the only non zero derivative is

$$g_{tt,x} = -ke^{kx}$$

The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right)$$

The non zero Christoffel symbols are

$$\Gamma^{x}_{tt} = \frac{1}{2} g^{xx} \left(-g_{tt,x}\right) = \frac{1}{2} \cdot (-1) \cdot -ke^{kx} = \frac{1}{2} ke^{kx}$$

The other are

$$\Gamma^{t}_{tx} = \Gamma^{t}_{xt} = \frac{1}{2} k$$

These are the required non zero Christoffel symbols. □

(c) A massive particle is instantaneously at rest ate $x = 0$. What is the instantaneous acceleration of the particle?
Solution:
The geodesic equation can be used to calculate the acceleration of the particle. From the geodesic equation we have

$$\frac{\partial U^{\mu}}{\partial \tau} = -\Gamma^{\mu}_{\alpha\beta} U^{\alpha} U^{\beta}$$

For particle at rest $v^i = 0$, $\implies$ $U^i = 0$. Using $U \cdot U = -1$ we get

$$(U^0)^2 g_{00} = -1 \qquad U^0 = e^{kx/2}$$

Since the only non zero Christoffel symbols are Γ^t_{tx} and Γ^x_{tt} we get

$$\frac{\partial U^x}{\partial \tau} = -\Gamma^x_{tt} U^0 U^0 = -\frac{1}{2} k e^{kx} e^{kx}$$

At the origin thus $x = 0$ we get

$$\frac{\partial U^x}{\partial \tau} = -\frac{1}{2} k$$

This gives the acceleration of the particle. □

(d) Compute the non-zero components of the Riemann tensor.
Solution:
And the Riemann tensor is given by

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\beta\mu}{}_{,\nu} + \Gamma^{\alpha}_{\beta\nu}{}_{,\mu}$$

calculating

$$\begin{aligned} R^{x}{}_{txt} &= \Gamma^{x}_{\sigma x}\Gamma^{\sigma}_{tt} - \Gamma^{x}_{\sigma t}\Gamma^{\sigma}_{tx} - \Gamma^{x}_{tx,t} + \Gamma^{x}_{tt,x} \\ &= -\Gamma^{x}_{tt}\Gamma^{t}_{tx} + \Gamma^{x}_{tt,x} \\ &= -\frac{1}{2}k \cdot \frac{1}{2} k e^{kx} + \frac{1}{2} k^2 e^{kx} \\ &= \frac{1}{4} k^2 e^{kx} \end{aligned}$$

Similarly the other component of Riemann tensor are

$$R^{t}{}_{xtx} = -\frac{1}{4} k^2$$

The other components are simply the cyclic permutation of the indices. □

(e) What are the non-zero terms in the Ricci Tensor and Ricci Scalar?
Solution:
The components of Ricci tensor in terms of elements of Riemann Tensor are

$$R_{\alpha\beta} = g^{\mu\nu} R^{\nu}_{\alpha\mu\beta}$$

Specifically for R_{tt} we get

$$\begin{aligned} R_{tt} &= g^{tt} \cancelto{0}{R^{t}_{ttt}} + g^{xx} R^{x}_{txt} \\ &= \frac{1}{4} k^2 e^{kx} \end{aligned}$$

Similarly the other component R_{xx} is

$$\begin{aligned} R_{xx} &= g^{tt} R^{t}_{xtx} + \cancelto{0}{g^{xx} R^{x}_{xtx}} \\ &= -e^{-kx} \cdot \frac{1}{4} k^2 e^{kx} \\ &= -\frac{1}{4} k^2 \end{aligned}$$

The Ricci scalar can be calculated by contracting the Ricci tensor as

$$R = R^t_t + R^x_x = g^{tt} R_{tt} + g^{xx} R_{xx} = -\frac{1}{4}k^2 - \frac{1}{4}k^2 = -\frac{1}{2}k^2 \tag{12.3}$$

So the Riccis scalar is $-1/2k^2$. □

(f) What is the Einstein tensor?
Solution:
The components of Einstein tensor are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{12.4}$$

The first component of this tensor is

$$G_{tt} = R_{tt} - \frac{1}{2} g_{tt} R = \frac{1}{4} k^2 e^{kx} + \frac{1}{2} e^{-kx} \cdot -\frac{1}{2} k^2 = 0$$

The other component is

$$G_{xx} = R_{xx} - \frac{1}{2} g_{xx} R = -\frac{1}{4} k^2 - \frac{1}{2} \cdot -\frac{1}{2} k^2 = 0$$

So the Einstein tensor is identically zero. □

12.5.2. In the generalized linear metric we found in class:

$$\begin{pmatrix} -1-2\psi & 0 & 0 & 0 \\ 0 & 1-2\phi & 0 & 0 \\ 0 & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{pmatrix}$$

where, for a non-relativistically moving source:

$$\nabla^2 = 4\pi(\rho + 3P); \qquad \nabla^2 \phi = 4\pi\rho$$

suppose you were in the interior of a spherically symmetric distribution with constant density and fixed equation of sate $w = -\frac{1}{3}$

(a) What is the acceleration on a test particle places a distance r from the center of the cloud. Would it fall inward or outward?
Solution:
Since ψ and ϕ are functions of r only we have non zero derivative of the components of metric only with respect to r. The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right)$$

The non zero Christoffel symbols are

$$\begin{aligned} \Gamma^t{}_{tr} = \Gamma^t{}_{rt} &= \frac{1}{2} g^{tt} \left(g_{tt,r} + \cancelto{0}{g_{rt,t}} - \cancelto{0}{g_{tr,r}} \right) \\ &= \frac{1}{2} \frac{-1}{1+2\psi} (-2\psi_{,r}) \\ &= \frac{\psi_{,r}}{1+2\psi} \end{aligned}$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^r{}_{tt} = \frac{\psi_{,r}}{1-2\phi} \qquad \Gamma^r{}_{rr} = -\frac{\phi_{,r}}{1-2\phi}$$

For a stationary particle $v^i = 0 \implies U^i = 0$. Using $U \cdot U = -1$ we get

$$(U^0)^2 g_{00} = -1 \implies U^0 = \sqrt{1+2\psi}$$

The geodesic equation can be used to calculate the acceleration of the particle. The geodesic equation is

$$\frac{\partial U^\mu}{\partial \tau} = -\Gamma^\mu_{\alpha\beta} U^\alpha U^\beta$$

The spatial acceleration of the particle is

$$\frac{\partial U^r}{\partial \tau} = \Gamma^r{}_{tt} U^0 U^0 = \frac{\psi_{,r}}{1-2\phi}(1+2\psi)$$

The quantity $\psi_{,r}$ can be calculated by using the fact that the Laplacian of ψ is given. In a spherically symmetric system $\nabla^2 \equiv \frac{\partial^2}{\partial r^2}$, so we get

$$\frac{\partial^2 \psi}{\partial r^2} = 4\pi(\rho + 3P)$$

Integrating once with respect to r we get

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = \psi_{,r} = 4\pi\rho\left(1 + 3\frac{P}{\rho}\right) r$$

Subsisting this in the expression for acceleration we get

$$\frac{\partial U^r}{\partial \tau} = \frac{\psi_{,r}}{1-2\phi}(1+2\psi) = 4\pi\rho(1+3w)r \cdot \frac{1+2\psi}{1-2\phi}$$

Given that $w = -\frac{1}{3}$ we get

$$\frac{\partial U^r}{\partial \tau} = 0 \cdot \frac{1+2\psi}{1-2\phi} = 0$$

So the radial acceleration of the particle is zero. Since for $i \neq r$, $U^i = 0$ and $\Gamma^i{}_{tt} = 0$ all other spatial components of acceleration is zero. So the spatial acceleration of the particle is identically zero. □

(b) What is the acceleration on a photon traveling perpendicular to the cloud also a distance r from the center. Would it be lensed inward or outward?

Solution:

Since the photon is trailing perpendicular to the cloud (in a straight line), we can assume (without loss of generality) the radial and azimuthal components of the velocity are zero, by choosing the direction of travel same as the radial coordinate. So, $U^\theta = 0, U^\phi = 0$ The spatial acceleration of the photon is

$$\frac{\partial U^r}{\partial \tau} = \Gamma^r{}_{tt} U^0 U^0 + \Gamma^r{}_{rr} U^r U^r$$

Subsisting the Christoffel symbols we get

$$\frac{\partial U^r}{\partial \tau} = \frac{\psi_{,r}}{1-2\phi}(U^0)^2 - \frac{\phi_{,}}{1-2\phi}(U^r)^2$$

Again by arguments of previous problem $\psi_{,r} = 0$, so we get

$$\frac{\partial U^r}{\partial \tau} = -\frac{\phi_{,r}}{1-2\phi}(U^r)^2$$

Again, in a spherically symmetric system $\nabla^2 \equiv \frac{\partial^2}{\partial r^2}$, we can similarly obtain $\phi_{,r}$ and ϕ by integrating the Laplacian of ϕ with respect to r once and twice respectively.

$$\phi_{,r} = 4\pi\rho r \qquad \phi = 2\pi\rho r^2$$

Since $\phi \ll 1$, $(U^r)^2 > 0$ and $\phi_{,r} = 4\pi\rho r > 0$ the final expression for the acceleration will turn out to be negative. Thus the acceleration would be inward and hence the photon will be lensed inward. □

12.5.3. **(Schutz 8.17)**

(a) A small planet orbits a static neutron star in a circular orbit whose proper circumference is 6×10^{11} m. The orbital period takes 200days of the planet's proper time. Estimate the mass M of the star.

Solution:

For the purpose of estimation we can assume that Newton's laws hold and that the time dilation effect is negligible. In that limit the proper time is just the time measured by observer. From Kepler's third law we have

$$t^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

If c is the circumference, it is given in terms of radius by, $c = 2\pi r$ subsisting c we get

$$t^2 = \frac{c^3}{2\pi GM} \implies M = \frac{1}{2\pi G}\frac{c^3}{T^2}$$

So for the given planet

$$t \approx \tau = 200days = 1.728 \times 10^7 s \qquad c = 6 \times 10^{11} m$$

So the mass is given by

$$M = \frac{1}{2\pi \cdot 6.672 \times 10^{-11}} \frac{(6 \times 10^{11})^3}{(1.728 \times 10^7)^2} = 1.726 \times 10^{30} kg$$

So the mass of the neutron star is 1.726×10^{30}kg. □

(b) Five satellites are placed into a circular orbit around a static black hole. The proper circumferences and proper periods of their orbits are given in a table below. Use the method of 12.5.3a to estimate the hole's mass. Explain the results you get for the satellites

circumference	2.5×10^6 m	6.3×10^6 m	6.3×10^7	3.1×10^8 m	6.3×10^9 m
proper period	8.4×10^{-3} s	0.055s	2.1s	23s	2.1×10^3 s

Solution:

Using the method of 12.5.3a we get

c(m)	t(s)	$\frac{1}{2\pi G}\left(\frac{c^3}{t^2}\right) kg$
2.5×10^6	8.4e-3	5.28×10^{32}
6.3×10^6	0.055	1.97×10^{32}
6.3×10^7	2.1	1.35×10^{32}
3.1×10^8	23	1.34×10^{32}
6.3×10^9	2.1×10^3	1.35×10^{32}

The obtained value for the mass seem to be converging towards $1.35 \times 10^{32} kg$ with the successive increase in the orbital circumference. So further away the satellite, the Newtonian approximation are more correct. □

12.5.4. **(Schutz 8.18)** Consider the field equation with cosmological constant. With Λ arbitrary and $k = 8\pi$.

(a) Find the Newtonian limit and show that we recover the motion of the planets only if $|\Lambda|$ is very small. Given the radius of Pluto's orbit is 5.9×10^{12} m, set an upper bound on $|\Lambda|$ from solar-system measurements
Solution:
The field equation is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Newtonian equation of motion is given by

$$\nabla^2 \phi = 4\pi\rho$$

specifically the first component of field equation

$$G_{00} = 8\pi T_{00} - \Lambda g_{00}$$

In the Newtonian limit, since $T_{00} = \rho$ and $g_{00} = -1$, I would expect in field equation term

$$\rho \to \rho + \frac{\Lambda}{8\pi}$$

I am assuming the limit to Λ comes from the maximum estimation of the mass density ρ in the solar system. Even if the space wre empty and only cosmological constant were present of that value, we would get the orbital radius of Pluto. So maximum value $\Lambda < \rho \times 8\pi$. The measured density of the solar system is in the order

$$\sim 1.3 \times 10^{-22} \frac{g}{cc} = 1.3 \times 10^{-19} \frac{kg}{m^3}$$

and so maximum Λ should be the same order. □

(b) By bringing Λ over th the RHS of Schutz eq 8.7 we can regard $-\Lambda g^{\mu\nu}/8\pi$ as the stress-energy tensor of 'empty space'. Given that he observed mass of the region of the universe near our Galaxy would have a density of about 1×10^{-27} kgm^3 if it were uniformly distributed, do you think that a value of $|\Lambda|$ near the limit you established in 12.5.4a could have observable consequences for cosmology? Conversely if Λ is comparable to the mass density of the universe, do we need to include it in the equations when we discuss the solar system?
Solution:
If Λ is is in the order as predicted in 12.5.4a, and the density of galaxy is in the order of $1 \times 10^{-27}\text{kg/m}^3$ then

$$\Lambda \gg \rho_{\text{galaxy}}$$

In that case $\rho \to \rho + \frac{\Lambda}{8\pi}$ would be dominated by Λ, so we would have to observable effect of the cosmological constant.
If the value of Λ is comparable to the density of the universe, then I would still assume that we would need to include in the calculation of solar system. □

12.5.5. **(Schuts 10.9)**

(a) Define a new radial coordinate in terms of the Schwarzschild r by

$$r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2.$$

Notice that as $r \to \infty, \bar{r} \to r$, while the event horizon $r = 2M$, where we have $\bar{r} = \frac{1}{2}M$. Show that the metric for spherical symmetry takes the form

$$ds^2 = -\left[\frac{1-2M/\bar{r}}{1+M/\bar{r}}\right]^2 dt^2 + \left[1+\frac{M}{2\bar{r}}\right]^4 \left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right]$$

Solution:
The Schwarzschild metric is

$$g_{\mu\nu} = \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & 1/(1-2M/r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The transformation of the elements of metric can be obtained by

$$g_{\bar{\mu}\bar{\nu}} = \Lambda^{\mu}_{\bar{\mu}}\Lambda^{\nu}_{\bar{\nu}} g_{\mu\nu}$$

where the elements of the transformation matrix $\Lambda^{\mu}_{\bar{\mu}}$ are given by

$$\Lambda^{\mu}_{\bar{\mu}} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}$$

Since the given metric is diagonal, the only non zero term in the metric $g_{\mu\nu}$ are with $\mu = \nu$. Expanding the metric transformation explicitly as a sum

$$g_{\bar{\mu}\bar{\nu}} = \Lambda^{\mu}_{\bar{\mu}}\Lambda^{\mu}_{\bar{\nu}} g_{\mu\mu}$$

Given the transformation $r \to \bar{r}(1+M/2\bar{r})^2$ and all other coordinates are unchanged we get

$$\begin{aligned} \Lambda^{r}_{\bar{r}} = \frac{\partial r}{\partial \bar{r}} &= \frac{\partial}{\partial \bar{r}}\left(\bar{r}\left(1+\frac{M}{2\bar{r}}\right)^2\right) \\ &= \left(1+\frac{M}{2\bar{r}}\right)^2 + 2\bar{r}\left(1+\frac{M}{2\bar{r}}\right)\left(-\frac{M}{2\bar{r}^2}\right) \\ &= \left(1+\frac{M}{2\bar{r}}\right)\left(1+\frac{M}{2\bar{r}}-\frac{M}{\bar{r}}\right) \\ &= \left(1+\frac{M}{2\bar{r}}\right)\left(1-\frac{M}{2\bar{r}}\right) \end{aligned}$$

for all other coordinates $\bar{t} = t, \bar{\phi} = \phi, \bar{\theta} = \theta$ so we get

$$\begin{aligned} &\Lambda^{\theta}_{\bar{\theta}} = \Lambda^{\phi}_{\bar{\phi}} = \Lambda^{t}_{\bar{t}} = 1 \\ &\Lambda^{\mu}_{\bar{\nu}} = 0 \text{ if } \mu \neq \nu \end{aligned}$$

Thus expanding the transformation of the metric explicitly we get

$$g_{\bar{t}\bar{t}} = \Lambda^{t}_{\bar{t}}\Lambda^{t}_{\bar{t}} g_{tt} = g_{tt} = -\left(1-\frac{2M}{r}\right)$$

Under the given transformation we have

$$1-\frac{2M}{r} = 1-\frac{2M}{\bar{r}\left(1+M/2\bar{r}\right)^2} = \frac{\bar{r}\left(1-M/2\bar{r}\right)^2 - 2M}{\bar{r}\left(1+M/2\bar{r}\right)^2} = \frac{\left(1-\frac{M}{2\bar{r}}\right)^2}{\left(1+\frac{M}{2\bar{r}}\right)^2} \tag{12.5}$$

So under the transformed coordinate system we get

$$g_{\bar{t}\bar{t}} = -\frac{(1 - M/2\bar{r})^2}{(1 + M/2\bar{r})^2}$$

The nexe component of the metric is

$$g_{\bar{r}\bar{r}} = \Lambda^r_{\bar{r}}\Lambda^r_{\bar{r}} g_{rr} = \left[\left(1 + \frac{M}{2\bar{r}}\right)\left(1 - \frac{M}{2\bar{r}}\right)\right]^2 \left(1 - \frac{2M}{r}\right)^{-1}$$

Using (12.5) we get in this expression we get

$$\begin{aligned} g_{\bar{r}\bar{r}} &= \left[\left(1 + \frac{M}{2\bar{r}}\right)\left(1 - \frac{M}{2\bar{r}}\right)\right]^2 \frac{(1 + M/2\bar{r})^2}{(1 - M/2\bar{r})^2} \\ &= \left(1 + \frac{M}{2\bar{r}}\right)^4 \end{aligned}$$

The next component of the transformed metric is

$$g_{\bar{\theta}\bar{\theta}} = \Lambda^\theta_{\bar{\theta}}\Lambda^\theta_{\bar{\theta}} g_{\theta\theta} = g_{\theta\theta} = r^2 = \bar{r}^2 \left(1 + \frac{M}{2\bar{r}}\right)^4$$

The final non zero component is

$$g_{\bar{\phi}\bar{\phi}} = \Lambda^\phi_{\bar{\phi}}\Lambda^\phi_{\bar{\phi}} g_{\phi\phi} = g_{\phi\phi} = r^2 \sin^2\theta = \bar{r}^2 \left(1 + \frac{M}{2\bar{r}}\right)^4 \sin^2\theta$$

Thus the final transformed metric is

$$g_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} -\frac{(1-M/2\bar{r})^2}{(1+M/2\bar{r})^2} & 0 & 0 & 0 \\ 0 & (1 + M/2\bar{r})^4 & 0 & 0 \\ 0 & 0 & \bar{r}^2 (1 + M/2\bar{r})^4 & 0 \\ 0 & 0 & 0 & \bar{r}^2 (1 + M/2\bar{r})^4 \sin^2\theta \end{pmatrix}$$

The line element in this metric is given by

$$ds^2 = -\left[\frac{1 - 2M/\bar{r}}{1 + M/\bar{r}}\right]^2 dt^2 + \left[1 + \frac{M}{2\bar{r}}\right]^4 \left[d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\phi^2\right] \tag{12.6}$$

Which is the required expression. □

(b) Define a quasi-Cartesian coordinates by the usual equations $x = \bar{r}\cos\phi\sin\theta$, $y = \bar{r}\sin\phi\sin\theta$, and $z = \bar{r}\cos\theta$ so that , $d\bar{r}^2 + \bar{r}^2 d\Omega^2 = dx^2 + dy^2 + dz^2$ Thus the metric has been converted into coordinates (x, y, z), which are called isotropic coordinates. Now take the limit as $\bar{r} \to \infty$ and show

$$ds^2 = -\left[1 - \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] dt^2 + \left[1 + \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] \left(dx^2 + dy^2 + dz^2\right)$$

Solution:
Under the transformation given

$$d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\phi^2 = dx^2 + dy^2 + dz^2$$

Under the limit $\bar{r} \to \infty$, the metric element $g_{\bar{t}\bar{t}}$ can be simplified

$$\begin{aligned} g_{\bar{t}\bar{t}} = \frac{(1 - M/2\bar{r})^2}{(1 + M/2\bar{r})^2} &= -\left(1 - \frac{M}{2\bar{r}}\right)^2 \left(1 + \frac{M}{2\bar{r}}\right)^{-2} \\ &= \left(1 - \frac{M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right)\left(1 - \frac{M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right) \\ &= \left(1 - \frac{M}{\bar{r}} - \frac{M}{\bar{r}} + \frac{M^2}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right) \\ &= \left(1 - \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right) \end{aligned}$$

Similarly under the approximation g_{rr}

$$g_{\bar{r}\bar{r}} = \left(1 + \frac{M}{2\bar{r}}\right)^4 = 1 + \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)$$

Subsisting this in the line element expression (12.6) we get

$$ds^2 = -\left[1 - \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] dt^2 + \left[1 + \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] \left(dx^2 + dy^2 + dz^2\right)$$

This is the required expression. □

(c) Compute the proper circumference of a circle at radius $\bar{r}$
Solution:
The circumference is given by the total distance traveled by a particle going at a constant distance $\bar{r}$ from the center, which is the length of the line under $\phi : 0 \to 2\pi$ The line element is

$$ds^2 = g_{\bar{\phi}\bar{\phi}} d\phi^2$$

So the total circumference is

$$C = \int_0^{2\pi} \sqrt{\bar{r}^2 \left(1 + \frac{M}{2\bar{r}}\right)^4} d\phi = 2\pi\bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2$$

Which is the proper circumference. □

(d) Compute the proper distance in traveling from $\bar{r}$ to $\bar{r} + d\bar{r}$.
Solution:
The line element is

$$ds^2 = g_{\bar{r}\bar{r}} d\bar{r}^2$$

The length going from $\bar{r} \to \bar{r} + d\bar{r}$ is

$$ds = \sqrt{g_{\bar{r}\bar{r}}} d\bar{r} = \sqrt{\left(1 + \frac{M}{2\bar{r}}\right)^4} d\bar{r} = \left(1 + \frac{M}{2\bar{r}}\right)^2 d\bar{r}$$

This gives the distance going from $\bar{r} \to \bar{r} + d\bar{r}$. □

12.6 Homework Six

12.6.1. **(Schutz 11.7)** A clock is in a circular orbit at $r = 10M$ in a Schwarzschild metric.

(a) How much time elapses on the clock during one orbit?
Solution:
The proper time and the interval are related by the expression $d\tau^2 = ds^2$. For circular orbit $dr = d\phi = 0$ so we get

$$d\tau^2 = ds^2 = g^{\phi\phi}(U_\phi)^2 d\phi^2 \implies d\tau = U^\phi d\phi$$

But for circular orbit the quantity $p_\phi = m\tilde{L}$ thus we obtain

$$U^\phi = \frac{1}{m}p^\phi = g^{\phi\phi}\frac{p_\phi}{m} = \frac{1}{r^2}\tilde{L}$$

The quantity $\tilde{L}^2 = \frac{Mr}{1-3M/r}$ substituting these we get

$$U^\phi = \frac{1}{r^2}\sqrt{\frac{Mr}{1-\frac{3M}{r}}}$$

The time elapsed is given by

$$\tau = \int_0^\tau d\tau = \int_0^{2\pi} \frac{1}{U^\phi} d\phi = \int_0^{2\pi} \sqrt{\frac{r^4(1-3M/r)}{Mr}} d\phi$$

Noting that, the integrand is independent of ϕ, for circular orbit at $r = 10M$ we obtain

$$\tau = 2\pi\sqrt{\frac{1000M^3}{M}\left(1 - \frac{3M}{10M}\right)} = 2\pi 10\sqrt{7}M$$

This is the time elapsed in the clock. □

(b) It sends out a signal to ta distant observer once each orbit. What time interval does the distant observer measure between receiving any two signals?
Solution:
The time elapsed for a distant observer is the coordinate time for the Schwarzschild metric. If it sends signal every orbit, the time elapsed for distant observer is the coordinate time for one full orbit. To find the coordinate time we have to get expression for $dt = f(\boldsymbol{x})d\phi$, where t is the coordinate time. From the definition of the ϕ component of four velocity

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = U^\phi = \frac{p^\phi}{m} = g^{\phi\phi}\frac{p_\phi}{m} = g^{\phi\phi}\tilde{L} = \frac{1}{r^2}\tilde{L}$$

Similarly from the $0th$ component of four velocity we get

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = U^0 = \frac{p^0}{m} = g^{00}\frac{p_0}{m} = g^{00}(-\tilde{E}) = \frac{\tilde{E}}{1-2M/r} \quad (12.7)$$

Combining these two we get

$$\frac{\mathrm{d}t}{\mathrm{d}\phi} = \frac{dt/d\tau}{d\phi/d\tau} = \left(\frac{r^3}{M}\right)^{1/2}$$

Now that we have obtained the functional form connecting the coordinate time and azimuthal angle. We can integrate to find

$$t = 2\pi\left(\frac{r^3}{M}\right)^{\frac{1}{2}}$$

For $r = 10M$ we obtain

$$t = 2\pi\sqrt{\frac{r^3}{M}} = 2\pi\sqrt{\frac{1000M^3}{M}} = 2\pi 10\sqrt{10}M. \tag{12.8}$$

This is the coordinate time that passes for one orbit which is the time measured by the distant observer and is also the time it elapses for distant observer for a complete revolution. □

(c) A second clock is located at rest at $r = 10$ next to the orbit of the first clock. How much time elapses on it between successive passes of the orbiting clock?
Solution:
The time is dilated in the orbiting clock by the time dilation factor which is simply

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{-1/g_{00}} = \sqrt{1 - 2M/r}$$

Now the proper time is given by

$$\tau = \sqrt{1 - \frac{2M}{r}}t$$

Substituting the coordinate time expression form (12.8) we get

$$\tau = \sqrt{1 - \frac{2M}{r}}2\pi\sqrt{\frac{r^3}{M}} \tag{12.9}$$

Substituting $r = 10M$ we obtain

$$\tau = 2\pi\sqrt{8}M.$$

This gives the time elapsed in the stationary clock as the clock makes one orbit. □

(d) Calculate (12.6.1b) again in seconds for an orbit at $r = 6M$ where $M = 14M_\odot$. This is the minimum fluctuation time we expect in the X-ray spectrum of Cyg X-1: why?
Solution:
For r = 6M substituting $r = 6M$ in (12.8) we get

$$t = 2\pi\sqrt{216}M = 12\pi\sqrt{6} \cdot 14M_\odot$$

The mass of sun $M_\odot = 1.9 \times 10^{30}kg = 1.476 \times 10^3 m$. Substation these

$$t = \frac{12\pi\sqrt{6} \cdot 1.476 \times 10^3}{3 \times 10^8} = 0.00636s = 6.36 \times 10^{-3}s$$

this is the time elapsed. □

(e) If the orbiting 'clock' is the twin Artemis, in the obit in (12.6.1d), how much does she age during the time her twin Diana lives 40years far from the black hole and at rest with respect to it?
Solution:
We already have for a circular orbit from (12.7) we have

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\tilde{E}}{1 - \frac{2M}{r}}$$

For a stable orbit in the Schwarzschild metric we have

$$\tilde{E} = \frac{1 - 2M/r}{\sqrt{1 - 3M/r}}$$

Substituting we get

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - 3M/r}}$$

Solving this differential equation we get

$$\int d\tau = \int \sqrt{1 - \frac{3M}{r}} dt$$

Setting $r = 6M$ gives

$$\tau = t\sqrt{\frac{1}{2}}$$

For $t = 40yr$ we get

$$\tau = \frac{40}{\sqrt{2}} = 28.28yr$$

This is the age of Artemis when her twin Diana lives 40yr. □

12.6.2. **(Schutz 11.21)** A particle of $m \neq 0$ falls radially toward the horizon of a Schwarzschild black hole of mass M. The geodesic it follows has $\tilde{E} = 0.95$

(a) Find the proper time required to reach $r = 2M$ from $r = 3M$.
Solution:
We have for a massive object the radial motion near the Schwarzschild metric satisfies:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)$$

The proper time is then given by

$$\tau = \int \frac{dr}{\sqrt{\tilde{E}^2 - 1 + \frac{2M}{r}}} \tag{12.10}$$

Making substituting $\alpha = \tilde{E}^2 - 1$ we get the following integral

$$\tau = \int \frac{dr}{\sqrt{\alpha + \frac{2M}{r}}}$$

The integral is

$$\tau = \left[\frac{2M\sqrt{r}}{\alpha\sqrt{2M + \alpha r}} - \frac{2M \operatorname{asinh}\left(\frac{\sqrt{2}\sqrt{\alpha}\sqrt{r}}{2\sqrt{M}}\right)}{\alpha^{\frac{3}{2}}} + \frac{r^{\frac{3}{2}}}{\sqrt{2M + \alpha r}}\right]_{3M}^{2M}$$

$$= \frac{3\sqrt{3}M^{\frac{3}{2}}}{\sqrt{3M\alpha + 2M}} - \frac{2\sqrt{2}M^{\frac{3}{2}}}{\sqrt{2M\alpha + 2M}} + \frac{2\sqrt{3}M^{\frac{3}{2}}}{\alpha\sqrt{3M\alpha + 2M}} - \frac{2\sqrt{2}M^{\frac{3}{2}}}{\alpha\sqrt{2M\alpha + 2M}} + \frac{2M \operatorname{asinh}\left(\sqrt{\alpha}\right)}{\alpha^{\frac{3}{2}}} - \frac{2M \operatorname{asinh}\left(\frac{\sqrt{6}\sqrt{\alpha}}{2}\right)}{\alpha^{\frac{3}{2}}}$$

Substituting $\alpha = 0.95^2 - 1$ we obtain

$$\tau = 1.1917M$$

This is the required time for the journey from $3M$ to $2M$ for a infilling particle. □

(b) Find the proper time required to reach $r = 0$ from $r = 2M$.
Solution:
Similar to previous part the proper time required is

$$\tau = \left[\frac{2M\sqrt{r}}{\alpha\sqrt{2M+\alpha r}} - \frac{2M\operatorname{asinh}\left(\frac{\sqrt{2}\sqrt{\alpha}\sqrt{r}}{2\sqrt{M}}\right)}{\alpha^{\frac{3}{2}}} + \frac{r^{\frac{3}{2}}}{\sqrt{2M+\alpha r}}\right]_{2M}^{0}$$
$$= \frac{2\sqrt{2}M^{\frac{3}{2}}}{\sqrt{2M\alpha+2M}} + \frac{2\sqrt{2}M^{\frac{3}{2}}}{\alpha\sqrt{2M\alpha+2M}} - \frac{2M\operatorname{asinh}\left(\sqrt{\alpha}\right)}{\alpha^{\frac{3}{2}}}$$

Substituting $\alpha = 0.95^2 - 1$ we obtain

$$\tau = 1.3745M$$

This is the required time for the journey from $2M$ to center for a infalling particle. □

(c) Find, on the Schwarzschild coordinate basis, its four-velocity components ate $r = 2.001M$.
Solution:
For radially moving object $U^\phi = U^\theta = 0$. The timelike component is given by

$$U^0 = -g^{00}\tilde{E} = \frac{\tilde{E}}{1-\frac{2M}{r}} = \frac{0.95}{1-\frac{2}{2.001}} = 1900.95$$

The radial component can be obtained by reusing (12.7) as

$$(U^r)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right) \qquad \Longrightarrow \qquad U^r = \sqrt{0.95^2 - 1 + \frac{2}{2.001}} = 0.949$$

Thus the four velocity is

$$U^\mu = \begin{pmatrix} 1900.95 \\ 0.949 \\ 0 \\ 0 \end{pmatrix}$$

This is the component of four velocity at $r = 2.001M$ □

(d) As it passes $2.001M$, it sends a photon out radially to a distant stationary observer. Compute the redshift of the photon when it reaches the observer.
Solution:
The energy observed by the distant observer is given by

$$E_{\text{obs}} = -U \cdot p = -\left(-U_0 p^0 + U_r p^r\right)$$

We can calculate the component p^r by using the fact that photon is massless. Since for photon we have

$$p^2 = -m^2 = 0$$

Expanding the dot product of the momentum we get

$$g_{tt}(p^t)^2 + g_{rr}(p^r)^2 = 0$$

But we have $p_t = E$ so we can rewrite this as

$$g^{tt}(p_t)^2 + g_{rr}(p^r)^2 = 0 \qquad \Longrightarrow p^r = \sqrt{-\frac{g^{tt}}{g_{rr}}}p_t = \sqrt{\frac{1-\frac{2M}{r}}{1-\frac{2M}{r}}}E = E$$

Substituting this in the observed energy expression we get

$$E_{\text{obs}} = (U_0 p^0 - U_r p^r) = \left(U^0 p_0 - U_r p^r\right) = E(U^0 - U_r)$$

But $U_r = g_{rr} U^r = \frac{U^r}{1-2M/r}$ substituting $U^0 = 1900.95$ and $U^r = 0.949$ we get

$$E_{\text{obs}} = E\left(1900.95 + \frac{0.949}{1 - \frac{2}{2.001}}\right) = 3801.95E$$

This gives the observed energy of the photon. So the redshift factor is simply

$$z = \frac{E_{\text{obs}} - E}{E} = \frac{3801.95E - E}{E} = 3800.95$$

Which is the required redshift factor. □

12.6.3. Using the relations that we derived in class:

$$a_{\text{y-stretching}} = \frac{2M}{r^3}\Delta y \qquad \text{and} \qquad a_{\text{x-compressing}} = \frac{M}{r^3}\Delta x$$

Throughout this problem, assume that you dropped from rest at infinity.

(a) Find the smallest black hole in which you could survive long enough to pass the event horizon.
Solution:
In the event horizon $r = 2M$. The maximum acceleration that human can survive is $a_{max} \sim 9g$. So we get

$$a_{\text{max}} = \frac{M}{(2M)^3}\Delta x \implies M = \sqrt{\frac{\Delta x}{4a_{\text{max}}}}$$

Substituting $\Delta x \sim 1m$ $g \sim 10\frac{m}{s^2}$ we get

$$M = \sqrt{\frac{1}{360}} = \frac{1}{6\sqrt{10}}s$$

Since $1s = 299792458m$ and $1m = 1.34 \times 10^{27}kg$ we get

$$M = \frac{1}{6\sqrt{10}} \cdot 299792458 \cdot 1.34 \times 10^{27} = 2.12 \times 10^{34}kg = 1.07 \times 10^4 M_\odot$$

This is the most massive black hole one can survive near the event horizon. □

(b) For a $1M_\odot$ black hole, how long does it take between the time you feel mildly uncomfortable (tidal force between head and feet is $2g$) and you die? This should be in proper time, of course.
Solution:
The tidal force will stretch so we have from the given stretching expression

$$a_{\text{y-stretching}} = \frac{2M}{r^3}\Delta y \implies r = \left(\frac{2M\Delta y}{a}\right)^{\frac{1}{3}}$$

For just being 'uncomfortable' $a = 2g$ gives

$$r = \left(\frac{2M_\odot \Delta y}{20}\right)^{\frac{1}{3}}$$

Substituting $M_\odot = 1.98 \times 10^{30}kg$ and $\Delta y \sim 0.5m$

$$r = 4.56 \times 10^9 (s^2 kg)^{1/3}$$

Substituting $1s = 299792458m$ and $1kg = 7.42 \times 10^{-28}m$ we get

$$r = 4.56 \times 10^9 (299792458^2 \cdot 7.42 \times 10^{-28})^{1/3} = 1.85 \times 10^6 m$$

For dying $a = 9g$ we get through similar process

$$r = 2.76 \times 10^9 (s^2 kg)^{1/3} = 2.76 \times 10^9 (299792458^2 \cdot 7.42 \times 10^{-28})^{1/3} = 1.12 \times 10^6 m$$

The proper time to travel between these two distance can be obtained by the expression as in Equation. (12.10) above

$$\tau = \int \frac{dr}{\sqrt{\tilde{E}^2 - 1 + \frac{2M}{r}}}$$

Here $\tilde{E}$ is proportional to initial energy for simplicity assuming $\tilde{E} = 1$ we get

$$\tau = \int \frac{1}{\sqrt{2M}} \sqrt{r} dr = \frac{1}{\sqrt{2M}} \frac{2}{3} r^{\frac{3}{2}} = \frac{\sqrt{2}}{3} \sqrt{\frac{r^3}{M}}$$

Proper time between these two distances is

$$\tau = \left[\frac{\sqrt{2}}{3} \sqrt{\frac{r^3}{M}} \right]_{r_1}^{r_2}$$

For $M = 1M_\odot$ we get

$$\tau = \left[\frac{\sqrt{2}}{3} \sqrt{\frac{r^3}{1.98 \times 10^{30}}} \right]_{1.85\times10^6}^{1.12\times10^6} = 4.45 \times 10^{-7} \left(\frac{m^3}{kg} \right)^{1/2}$$

Substituting $1kg = 7.42 \times 10^{-28}m$ and $1m = \frac{1}{299792458}s$ we get

$$\tau = 4.45 \times 10^{-7} \left(1.34 \times 10^{27} m^2\right)^{\frac{1}{2}} = 4.45 \times 10^{-7} \cdot 1.22 \times 10^5 s = 5.44 \times 10^{-2} s$$

This gives the time for mild uncomfortably and death. □

(c) How about a $10M_\odot$
Solution:
Repeating the same process for $M = 10M_\odot$ we get

$$r_1 = 9.83 \times 10^9 \left(s^2 kg\right)^{1/3} = 3.98 \times 10^6 m$$
$$r_2 = 5.95 \times 10^9 \left(s^2 kg\right)^{1/3} = 2.41 \times 10^6 m$$
$$\tau = 4.44 \times 10^{-7} \left(\frac{m^3}{kg} \right)^{1/3} = 5.426 \times 10^{-2} s$$

So for a $10M_\odot$ the time interval for the falling person from mild uncomfortability to death is $5.42 \times 10^{-2} s$. □

12.6.4. **(Schuts 12.9)**

(a) Show that a photon which propagates in a radial null geodesic of the metric, , has energy $-p_0$ inversely proportional to $R(t)$.

Solution:
The given metric is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & R^2(t)r^2 & 0 \\ 0 & 0 & 0 & R^2(t)r^2\sin^2\theta \end{pmatrix}$$

For radial geodesic $U^\phi = U^\theta = 0$. Since photon is massless we get

$$p \cdot p = 0 \implies g^{00}(p_0)^2 + g^{rr}(p_r)^2 = 0$$

Simplifying gives

$$(p_0)^2 = -\frac{g^{rr}}{g^{00}}(p_r)^2 = \frac{R^2(t)}{1-kr^2}(p_r)^2 \tag{12.11}$$

We now have to find the relationship between p_r and the element of metric. The next relationship comes from the geodesic equation as

$$\dot{p}^\mu = \Gamma^\mu_{\rho\nu}\frac{p^\rho p^\nu}{p^0}$$

Specifically for $\mu = 0$ we get

$$\dot{p}^0 = \Gamma^0_{\rho\nu}\frac{p^\rho p^\nu}{p^0}$$

We need the Christoffel symbols for this. The Christoffel symbols are given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right)$$

The only required Christoffel symbols are $\Gamma^0_{\alpha\beta}$

$$\Gamma^0_{\nu\rho} = \frac{1}{2}g^{00}\left(g_{\nu 0,\rho} + g_{\rho 0,\nu} - g_{\nu\rho,0}\right)$$

Explicitly

$$\Gamma^0_{rr} = \frac{1}{2}g^{00}\left(g_{r0,r} + g_{r0,r} - g_{rr,0}\right) = \frac{1}{2}(-1)\left(-\partial_t\left(\frac{R^2(t)}{1-kr^2}\right)\right) = \frac{\dot{R}(t)R(t)}{1-kr^2}$$

Substituting this in the geodesic equation we get

$$\dot{p}^0 = -\frac{\dot{R}(t)R(t)}{1-kr^2}\frac{p^r p^r}{p^0}$$

But from (12.11) we have $(p^r)^2$ and substituting we get

$$\dot{p}^0 = -\frac{\dot{R}(t)R(t)}{1-kr^2}\frac{(1-kr^2)(p^0)^2}{R^2(t)p^0}$$

$$\dot{p}^0 = -\frac{\dot{R}(t)}{R(t)}p^0$$

This is a differential equation , solving we get

$$\frac{dp^0}{p^0} = -\frac{dR(t)}{R(t)} \qquad \ln\left(p^0\right) = -\ln(R(t)) \implies p^0 \propto \frac{1}{R(t)}$$

Lowering the index of p^0 in the LHS we get

$$p_0 = g_{00}p^0 = -p^0 \implies p_0 \propto -\frac{1}{R(t)}$$

Which is the required expression. □

(b) Show from this that a photon emitted at time t_e and received a time t_r by observers at rest in the cosmological reference from is redshift ed by

$$1 + z = \frac{R(t_r)}{R(t_e)}$$

Solution:
For an observer at rest $v^i = 0 \implies U^i = 0$. Using $U \cdot U = -1$ gives

$$g_{00}(U^0)^2 = -1 \implies U^0 = \sqrt{-\frac{1}{g_{00}}} = 1$$

thus the observed energy is

$$E_{\text{obs}} = -p \cdot U_{\text{obs}} = -p_0 U^0 = -p_0$$

calculating the redshift we get

$$z = \frac{E_{\text{obs}}(t_e) - E_{\text{obs}}(t_r)}{E_{\text{obs}}(t_r)} = \frac{-\frac{1}{R(t_e)} + \frac{1}{R(t_r)}}{-\frac{1}{R(t_r)}} =$$

Simplifying

$$1 + z = 1 + \frac{-\frac{1}{R(t_e)} + \frac{1}{R(t_r)}}{-\frac{1}{R(t_r)}} = \frac{R(t_r)}{R(t_e)}$$

This is the required expression. □

12.6.5. **(Schuts 12.20)** Assume that the universe is matter dominated and find the value of ρ_Λ that permits the universe to be static.

(a) Because the universe is matter-dominated at the present time, we can take $\rho_m(t) = \rho_0 \left[\frac{R_0}{R(t)}\right]^3$ where the subscript 0 refers to the static solution we are looking for. Differentiate the 'energy' equation

$$\frac{1}{2}\dot{R}^2 = -\frac{1}{2}k + \frac{4}{3}\pi R^2 (\rho_m + \rho_\Lambda) \tag{12.12}$$

with respect to time to find the dynamical equation governing a matter dominated universe:

$$\ddot{R} = \frac{8}{3}\pi\rho_\Lambda R - \frac{4}{3}\pi\rho_0 R_0^3 R^{-2}$$

Set this to zero to find the solution

$$\rho_\Lambda = \frac{1}{2}\rho_0$$

For Einstein's static solution, the cosmological constant energy density has to be half of th matter energy density.

Solution:
As instructed, differentiating with respect to tie we get

$$\frac{1}{2} \cdot 2 \cdot \ddot{R}\dot{R} = \frac{8}{3}\pi R\dot{R}(\rho_m + \rho_\Lambda) + \frac{4}{3}\pi R^2 \left(\dot{\rho}_m + \dot{\rho}_\Lambda\right)$$
$$\ddot{R} = \frac{8}{3}\pi R(\rho_m + \rho_\Lambda) + \frac{4}{3}\pi R^2 \left(\dot{\rho}_m + \dot{\rho}_\Lambda\right)$$

But the functional form of $\rho_m(t)$ is given differentiating we get

$$\dot{\rho}_m = -3\frac{R_0^3 \rho_0 \dot{R}}{R^4}$$

And for matter dominated universe $\dot{\rho}_\Lambda = 0$ substituting these

$$\ddot{R}\dot{R} = \frac{8}{3}\pi R\dot{R}\left(\rho_m + \rho_\Lambda\right) - 4\pi\frac{\rho_0 R_0^3 \dot{R}}{R^2}$$
$$\ddot{R} = \frac{8}{3}\pi R\left(\rho_m + \rho_\Lambda\right) - 4\pi\frac{\rho_0 R_0^3}{R^2}$$

Which is the required dynamical equation. At current time we have $R = R_0$ so we get

$$\ddot{R} = \frac{8}{3}R_0\left(\rho_0 + \rho_\Lambda\right) - 4\rho_0 R_0 = \frac{8}{3}R_0\rho_\Lambda - \frac{4}{3}R_0\rho_0$$

Setting this equal to zero we get

$$\frac{8}{3}R_0\rho_\Lambda = \frac{4}{3}R_0\rho_0$$

We obtain

$$\rho_\Lambda = \frac{1}{2}\rho_0$$

This is the required expression. □

(b) Put our expression for ρ_m into the right-hand-side of (12.12) to gen an energy-like expression which has a derivative that has to vanish for a static solution. Verify that the above condition of ρ_Λ does indeed make the first derivative vanish.
Solution:
Substituting ρ_m we obtain

$$\frac{1}{2}\dot{R}^2 = -\frac{1}{2}k + \frac{4}{3}\pi R^2\left(\frac{\rho_0 R_0^3}{R^3} + \frac{1}{2}\rho_0\right)$$

For static solution the second term on the right has to have vanishing derivative because the first being constant has zero derivative already. Checking

$$\frac{\partial}{\partial R}\left[\frac{4}{3}\pi R^2\left(\frac{\rho_0 R_0^3}{R^3} + \frac{1}{2}\rho_0\right)\right] = \frac{4}{3}\pi\rho_0\frac{\partial}{\partial R}\left[\left(\frac{R_0^3}{R} + \frac{R^2}{2}\right)\right] = \frac{4}{3}\pi\rho_0\left[\left(-\frac{R_0^3}{R^2} + R\right)\right]$$

For initial time we have $R = R_0$ this expression evaluates to zero. □

(c) Compute the second derivative of the right-hand-side of (12.12) with respect to R and show that, the static solution, it is positive. This means that the 'potential' is a minimum and *Einstein's static solution is stable.*

Solution:
The second derivative is

$$\frac{4}{3}\pi\rho_0\frac{\partial}{\partial R}\left[\left(-\frac{R_0^3}{R^2}+R\right)\right]=\frac{4}{3}\pi\rho_0\left[\left(2\frac{R_0^3}{R^3}+1\right)\right]$$

For today $R = R_0$ and we get

$$\frac{4}{3}\pi\rho_0\left[\left(2\frac{R_0^3}{R_0^3}+1\right)\right]=\frac{4}{3}\pi\rho_0\left[\left(2\frac{R_0^3}{R_0^3}+1\right)\right]=4\pi\rho_0$$

For $\rho > 0$ the second derivative is positive. This means the solution is stable. □