Uniformly Accelerated Motion

In the case of uniformly accelerated motion (in one dimension), with acceleration a, we know that a particle's position x and velocity v are given by

$$\begin{aligned} x &= x_0 + v_0\, t + \tfrac{1}{2}\, a\, t^2 \\ v &= v_0 + a\, t \end{aligned}$$

These equations represent a mathematical connection between the initial (subscript 0) state of the system and the system at any subsequent time t.

The basic idea behind any *numerical* algorithm for determining the particle trajectory is break the motion into many small segments, or *time steps*. In the numerical world, the continuous trajectory $\{x(t), 0 \le t < T\}$ is represented by a discrete set of points $\{x(t_i), i = 0, ..., N\}$, with $t_0 = 0, t_N = T$. Strictly speaking, we know *nothing* about the behavior of the system at intermediate times. We normally interpolate between data points as needed, but realize that this in fact entails making certain (reasonable) assumptions about the mathematical form of the true solution.

Given the state of the system at time t_0 (i.e. the initial conditions of the calculation), the goal of our algorithm is to determine the state of the system at any subsequent time t_i. In most cases, this boils down to the following:

> given $x_i = x(t_i)$ at time t_i, and a time step δt_i, determine a numerical estimate of the state of the system at time $t_{i+1} = t_i + \delta t_i$.

In other words, we must write down a rule that maps (x_i, v_i) into (x_{i+1}, v_{i+1}). Such a rule is known as an *integration scheme*. Clearly, applying the scheme repeatedly will accomplish the desired goal of advancing the system arbitrarily far forward in time. The result of each step becomes the initial state for the next.

For uniformly accelerated motion, this is easy to do, as the formula given above connects the positions and velocities at the start and end of the step. We can write

$$\begin{aligned} x_{i+1} &= x_i + v_i \delta t + \tfrac{1}{2} a \delta t^2\,, \\ v_{i+1} &= v_i + a \delta t\,, \\ t_{i+1} &= t_i + \delta t\,. \end{aligned}$$

This is our first integration scheme! The mapping from state i to state $i+1$ is the discrete numerical analog of the continuous integral operator that maps the initial conditions of the problem (at $t = 0$) into the state of the system at time t.

Note that the map is *explicit*—the procedure for getting from state i to state $i+1$ depends only on quantities determinable at time t_i. This statement is the key operational difference between the numerical scheme and the previous analytic expression.

Notice, by the way (as we'll see in Homework 3) that the integration scheme just described is *exact* for the special case of uniformly accelerated motion—not too surprising, considering how it was derived! In general, however, the scheme produces only an *approximation* to the correct solution, as we will explore in future exercises.