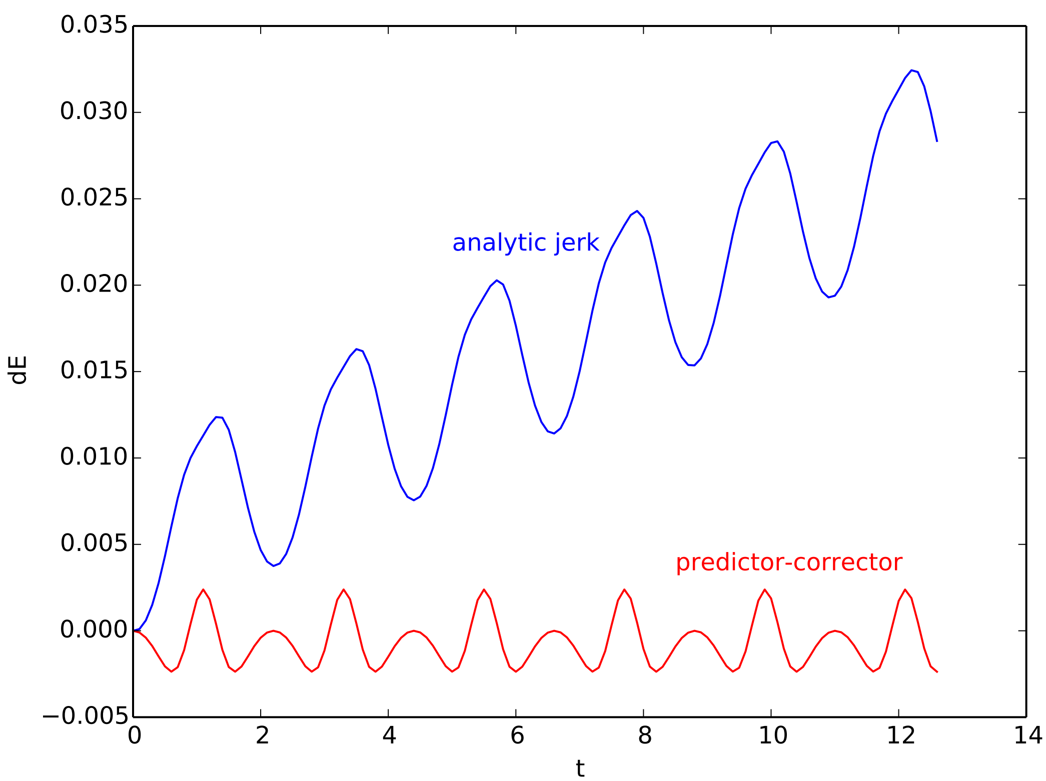
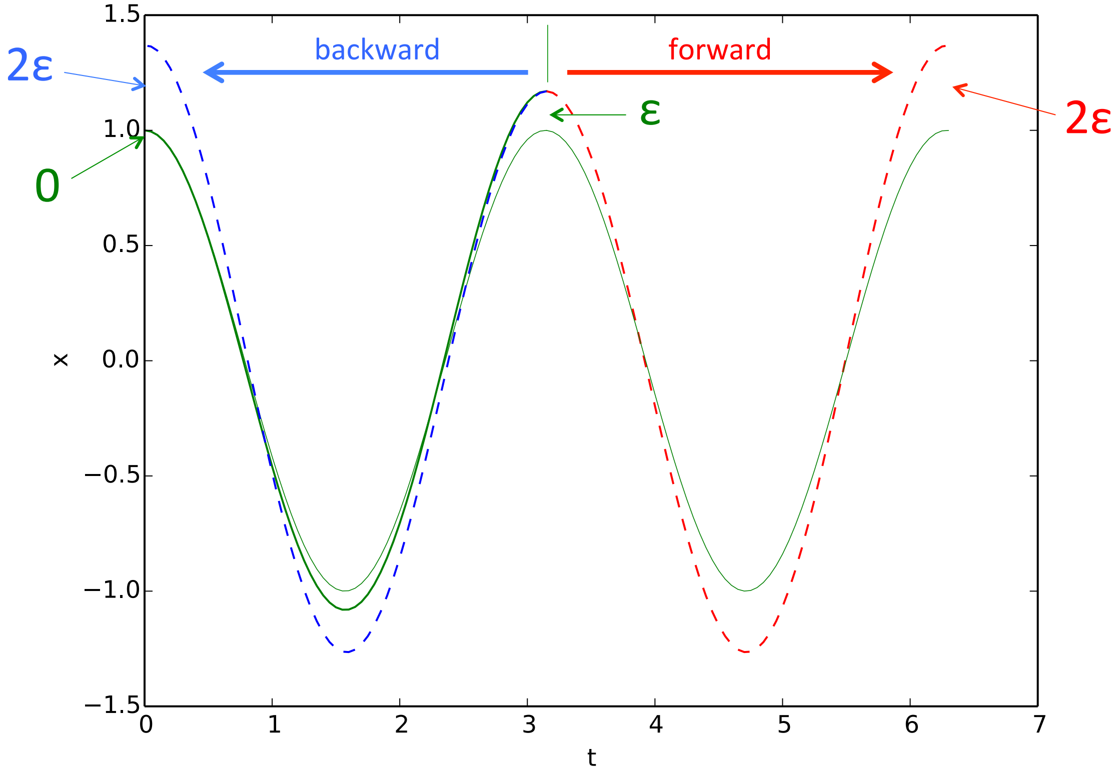
Time Reversibility and Energy Conservation

We have seen both experimentally and analytically that the "numerical jerk" second-order integration scheme—the predictor–corrector scheme—is time reversible. We can take as many steps as we like in the forward direction, then take the same number of steps in the reverse direction and end up at *precisely* the initial position. If we do this with the "analytic jerk" method, we find that it returns only approximately to its starting point, making an error that scales as the second power of the time step.

We can prove algebraically that the predictor–corrector scheme is time-reversible (Homework 4). Now let's explore some consequences of this fact. If we plot the time-dependence of the energy error for the two second-order schemes, as applied to the nonlinear oscillator, we discover a remarkable fact—the error in the analytic method grows steadily over time, while the error in the predictor–corrector schemes returns exactly to *zero* at the end of each oscillation. This means that the predictor–corrector scheme makes *no* long-term error, averaged over an integral number of periods!



Why is this? In fact, we can understand it quite easily. Suppose that, integrating forward for one period, we make an energy error of ϵ, as indicated in the figure below. Integrating forward for another period (red dashed line), we make another error ϵ, for a total of 2ϵ. The symmetry of the problem implies says that integrating forward for one period is equivalent to integrating backwards, so if we reversed our step and returned to our starting point after the first period (blue dashed line) we would expect to incur a additional error of ϵ, again for a total of 2ϵ. But time symmetry says that the total error must be zero on return to the start. Hence $2\epsilon = 0$ and energy is precisely conserved.



Time reversibility is important because it guarantees that energy (or *any* conserved quantity) is conserved numerically over the course of a periodic orbit. Thus we can use the predictor–corrector scheme to integrate the motion forward in time for hundreds or even millions of time units without seeing any growth in the energy error.

Note that, as we vary the time step, the *amplitude* of the energy error will scale as the square of the step, since the method is second order, but we will never see any long-term growth.