Quadratic Interpolation Near a Maximum

Linear interpolation works well in many circumstances, but not in the vicinity of a maximum or minimum, where the function is clearly not well approximated by a straight line. Can we improve on simply picking the maximum point as our estimate of the true maximum?

Let's imagine that we have detected a maximum in a numerically defined function, as discussed in class. In the terms of the example on the main web page, that means that we know the points (t_{pp}, x_{pp}), (t_p, x_p), and (t, x), where $t - t_p = t_p - t_{pp} = \Delta t$. Since linear interpolation is inappropriate, we instead consider a quadratic fit to the function in the neighborhood of the maximum. Since t_p is our best estimate of the maximum, let's write the fit as

$$x(\tau) = a + b(\tau - t_p) + c(\tau - t_p)^2.$$

We can obtain a, b, and c by requiring that the fit passes through the three points just listed:

$$\begin{aligned} x_{pp} &= a + b(-\Delta t) + c(\Delta t^2) \\ x_p &= a \\ x &= a + b(\Delta t) + c(\Delta t^2). \end{aligned}$$

These equations are easily solved to find

$$\begin{aligned} a &= x_p \\ b &= \frac{x - x_{pp}}{2\Delta t} \\ c &= \frac{x - 2x_p + x_{pp}}{2\Delta t^2}, \end{aligned}$$

so the maximum occurs at

$$\begin{aligned} \tau &= t_p - \frac{b}{2c} \\ &= t_p - \frac{x - x_{pp}}{2(x - 2x_p + x_{pp})}\,\Delta t, \end{aligned}$$

and the maximum value is

$$\begin{aligned} x_{max} &= a - \frac{b^2}{4c} \\ &= x_p - \frac{(x - x_{pp})^2}{4(x - 2x_p + x_{pp})}. \end{aligned}$$