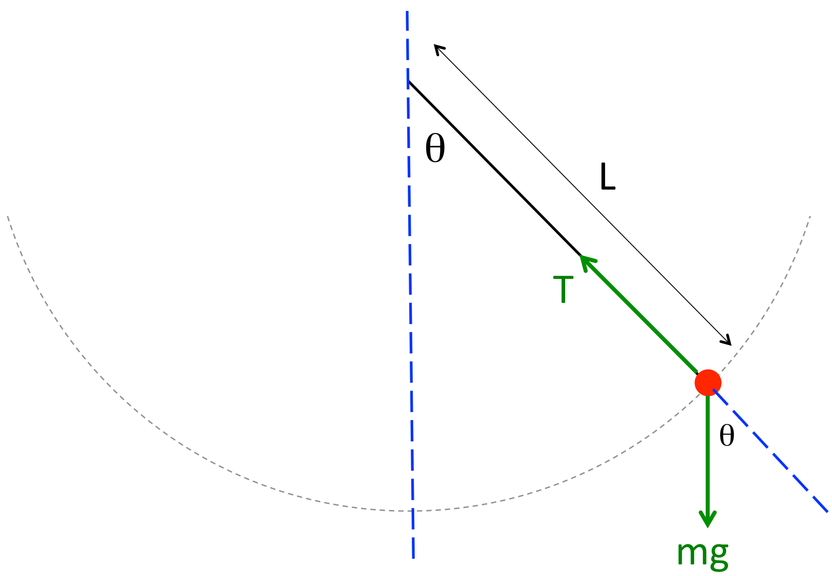
The Nonlinear Pendulum

Consider a pendulum consisting of a mass m fixed to one end of a rigid rod of negligible mass and length L. The rod is free to rotate in a vertical plane about the other end, so the mass is constrained to move on a circle, as illustrated in the diagram below. At any instant, the rod makes an angle θ with the vertical.



We can understand the motion by drawing a free-body diagram of the system and resolving the forces acting parallel and perpendicular to the rod. Since the particle moves on a circle, the forces in the radial direction must combine to provide the correct centripetal force, $m\omega^2 L$, where

$$\omega = \frac{d\theta}{dt} \equiv \dot{\theta}.$$

Hence, at any instant, the tension T in the rod must be

$$T = m\omega^2 L + mg\cos\theta.$$

In the transverse direction, the net force is simply $-mg\sin\theta$, where we count a force positive is it acts in the direction of *increasing* θ. This force must be equal to the mass times the transverse acceleration of the particle, $mL\ddot{\theta}$, so we find

$$\ddot{\theta} = -\frac{g}{L}\sin\theta.$$

This is the equation of motion of the pendulum. It is convenient to write $k = g/L$, so the equation becomes

$$\ddot{\theta} = -k\sin\theta.$$

This is a nonlinear equation, and there is no analytic solution, although numerical solution is straightforward, as we will see. The conserved energy of the motion is

$$E = \tfrac{1}{2}v^2 + k(1 - \cos\theta).$$

Notice that for small θ we can write

$$\sin\theta \approx \theta,$$

in which case the motion reduces to simple harmonic motion, as discussed previously in class and in PHYS 113/4. Thus we can write down an approximate solution for small θ, but we should expect the true solution to deviate from this result when θ is large.